New Works and New Editions

By the Rev. H. GOODWIN. M.A.,

Late Fellow of Gonville and Caius College, and Minister of St. Edward's, Cambridge.

An ELEMENTARY COURSE of MATHEMATICS, designed principally for Students of the University of Cambridge. Fourth Edition. 8vo. 15*s.*

*** This Edition contains 100 *pages of additional matter*, and the price has been reduced from 18*s.* to 15*s.*

A COLLECTION of PROBLEMS and EXAMPLES, adapted to the "Elementary Course of Mathematics." With an Appendix, containing the Questions proposed during the first Three Days of the Senate House Examinations in the Years 1848, 1849, 1850, and 1851. Second Edition. 8vo. 6*s.*

ASTRONOMY, Elementary Chapters in, from the "Astronomie Physique" of Biot. 8vo. 3*s.* 6*d.*

PARISH SERMONS. First and Second Series. 12mo. 6*s.* each.

SHORT SERMONS at the Celebration of the Lord's Supper. 12mo. 4*s.*

SOLUTIONS of GOODWIN'S Collection of PROBLEMS and EXAMPLES. By the Rev. W. W. HUTT, M.A., Fellow and Sadlerian Lecturer of Gonville and Caius College. 8vo. 8*s.*

The GREEK TESTAMENT: with a Critically Revised Text; a Digest of various Readings; Marginal References to Verbal and Idiomatic Usage; Prolegomena; and a Critical and Exegetical Commentary. For the use of Theological Students and Ministers. By H. ALFORD, M.A., Vicar of Wymeswold, Leicestershire, and late Fellow of Trinity College, Cambridge. In 3 vols.

Vol. I. Containing the Four Gospels, £1. 4*s.*

Vol. II. Containing the Acts of the Apostles, the Epistles to the Romans and Corinthians, £1. 4*s.*

Vol. III. Pre[redacted]

ELEMENTARY MECHANICS

DESIGNED CHIEFLY

FOR THE USE OF SCHOOLS.

BY

HARVEY GOODWIN, M.A.,

LATE FELLOW AND MATHEMATICAL LECTURER
OF GONVILLE AND CAIUS COLLEGE.

1851.

ELEMENTARY MECHANICS

DESIGNED CHIEFLY FOR THE USE OF SCHOOLS.

PART I.

STATICS.

BY

HARVEY GOODWIN, M.A.,

LATE FELLOW AND MATHEMATICAL LECTURER
OF GONVILLE AND CAIUS COLLEGE.

CAMBRIDGE: JOHN DEIGHTON.
LONDON: SIMPKIN, MARSHALL & CO., AND GEORGE BELL.
LIVERPOOL: DEIGHTON & LAUGHTON.
M.DCCC.LI.

Cambridge:
Printed at the University Press.

PREFACE.

Galileo is accused of having adopted the conversational form in his physical treatises, in order that he might be the more free under the disguise of his interlocutors to praise his own discoveries. Without caring to undertake a complete defence, I can easily suggest a more charitable reason for his having expounded his views in the form of a dialogue. Galileo might feel, indeed must have felt, a freedom of explanation, and a facility in putting difficulties and solving them, when he adopted the dialogue, of which the greater stiffness and dignity of an ordinary treatise might not seem to allow: and certainly we must admit that he has been singularly happy in bringing out and explaining the various points of his subject in an easy and entertaining manner.

It was probably the example of Galileo's dialogues, which suggested to me the possible advantage of introducing the conversational element into school-books on Mechanics. I knew, however, that a mere conversation-book could not be made to convey the subject in a form such as the student requires; I knew that the fundamental propositions must be given in that plain stern form in which they are usually presented; yet I thought that the dialogue might be introduced as subsidiary to the common method; and, finally, I determined to attempt an elementary treatise founded upon the union of the two. The following book, therefore, has been composed upon

this principle; first, there is a general introductory conversation between the tutor and his pupil upon the subject of Mechanics; then follow the various chapters of the book in order, and to each chapter (except the last, which contains problems only) is appended a conversation longer or shorter, as the case may be, containing explanations of difficulties, collateral matter, and the like, such as I deemed it inconvenient to introduce into the body of the chapter; and then follows an examination paper. It is intended that each chapter should be complete in itself, the appended conversation being merely supplementary and explanatory. And I may mention that I regard the conversations as a depôt for illustrative matter, and that I shall feel thankful for any hints which may tend in future editions to render this part of the work more useful.

There is another feature in which this treatise differs from any which have preceded it. I have divided the subject into *experimental* and *demonstrative* Mechanics; while I have felt the extreme importance of putting Mechanics upon its true basis as a demonstrative science, I have, nevertheless, considered that it may be treated experimentally, and that an experimental introduction is probably (for young minds) the simplest and the best. I have endeavoured, therefore, by means of two chapters depending upon experiment, to familiarize the student with the general notions of the subject before proceeding to its more abstract treatment.

These are the only points in the plan of the Work which I think it necessary to notice; the utility of the plan, and the manner of its execution, I must leave to the judgment and experience of those teachers and students by whom the book may be used: I will only add, that, as

I cannot expect to earn fame in such a humble walk of mathematics, I hope I shall have the credit of an honest endeavour to supply a want which I know to be very generally felt; and further, I shall feel obliged if those who use my book, and approve it on the whole, will favour me with any suggestions which may occur to them for the improvement of its details.

The treatise on Dynamics is not yet prepared; I am anxious, before it shall be published, to see what view is taken of the plan which I have adopted for the Statics.

I may also take this opportunity of mentioning, that as in the treatise on Dynamics I shall have frequent occasion to refer to the properties of the Conic Sections, I have thought it desirable to prepare a treatise on that subject, adapted as much as possible to the use of Schools. This treatise may be expected shortly.

H. GOODWIN.

CAMBRIDGE,
November, 1851.

TABLE OF CONTENTS.

INTRODUCTORY CONVERSATION.

Tutor. We have now laid so good a foundation of Geometry, Algebra, and Trigonometry, that I think we may safely proceed to an application of our Mathematics to the subject of *Mechanics.*

Pupil. I have heard Mechanics spoken of as a difficult subject, and I was afraid that it would be quite out of my reach at present.

T. A difficult subject no doubt it is, taken in all its length and breadth; for the whole theory of the motion of the earth, moon and planets, not to mention others still more difficult, comes under the head of Mechanics. And in order to solve the various beautiful mechanical problems which nature presents to us, mathematicians have been led to devise the most refined methods of calculation; so that in studying the physical investigations of the great masters of science, one hardly knows which to admire more, the beautiful order of creation which their labours have explained, or the fine intellect with which mankind have been endowed and which has rendered such explanations possible. But without aspiring at present to such high regions of science, for which neither your age nor your present acquirements in pure mathematics fit you, it may be very possible for you by attention and care to acquire a knowledge, and a very useful knowledge too, of the fundamental principles of Mechanics.

P. What do you mean by the term *pure* Mathematics?

T. I mean that great branch of Mathematics, which treats only of the pure conceptions of number, space, and quantity. Arithmetic, Geometry, and Algebra, may be con-

sidered as the three fundamental subjects in pure mathematics; Trigonometry, as usually treated in the present day, is a compound of Algebra and Geometry; and all the higher branches of pure mathematics, with which I hope you will one day be acquainted, may be considered as belonging to one or both of these.

P. What term do you apply to those branches of Mathematics which are not included in this class?

T. We generally make use of the term *Mixed* Mathematics. In this class are included Mechanics, Optics, Hydrostatics, Astronomy, the theories of Sound, Light, Heat, Electricity, and the like. You will see at once that under these heads we have to deal with conceptions much more difficult than in the pure sciences; no boy of ordinary ability has any difficulty in conceiving of a triangle, a square, a circle, or in comprehending what is meant by a straight line, although Mathematicians have had some doubt as to the best mode of defining it, or in acquiring the rules of Arithmetic, or in solving common Algebraical problems; but the question is a very different one when we come to the consideration of such a conception as that of Force, which is the foundation of Mechanics; a triangle or a straight line can be represented on paper, and number can be illustrated by divers familiar examples, thus the addition of 2 and 3 is immediately understood by the familiar notion of a boy having 2 oranges and 3 more being given to him; but Force can be by no means so easily illustrated, we know it by its effects, and it is not at all easy to see how those effects can be made the subject of calculation. Moreover, all the mixed sciences are more or less experimental sciences; that is, we ascertain certain laws or facts by experiment or observation, and we then make use of mathematical science for the purpose of ascertaining the results of those observed laws; and it is probably from this mixture of abstract calculation with experimental truth that the name of mixed mathematics arises. The superior difficulty of the mixed sciences may be argued

also from this, that the acute Greeks, who succeeded so admirably in Geometry, made, comparatively speaking, little progress in those branches of mixed mathematics to which they applied themselves.

P. And you think that I have made sufficient progress in pure mathematics to advance to the more formidable science of Mechanics?

T. I think so: I should not recommend the step, if I were not satisfied that you had thoroughly laid hold upon the principles of the subjects which you have read, and that you were able to apply them. You will remember how careful I have always been that you should not merely learn a certain number of propositions, and that I have insisted upon a very extensive application of what you have read to examples and easy problems; by this means I trust you will find that you have realized the end which I have had ever in view, namely, that your mathematical knowledge should not be a weight for you to carry, but a tool for you to use with ease and dexterity.

P. I confess that I have often a little murmured in my mind at your inexorable determination concerning examples and problems.

T. That confession does not surprise me; but you will now begin to enjoy the sweets of your labour, you will find that you will be able to apply your previous knowledge to very interesting mechanical problems with comparative ease, having only the difficulties peculiar to your subject to contend with; whereas if while you required all your attention to enable you to grapple with the mechanical principles of a problem, you had your mind distracted by the difficulty of managing an undisciplined regiment of x's and y's, sines, cosines, and tangents, you would experience a degree of mental confusion very prejudicial to your advance in mechanical science.

I will now explain to you in a few words the plan according to which I propose to teach you something of Mechanics. We will first take a chapter of an elementary

treatise, such as this which I have prepared for you, and now hold in my hand, and we will read it carefully*; then you shall ask me to explain any difficulties which may present themselves, or perhaps I may suggest difficulties to you and questions which may serve to test the manner in which you have comprehended what we have read; and having thus discussed the subject of the chapter in a familiar manner, I will leave you a paper of written questions or of problems and examples, upon which you may at your leisure try your skill; and until this paper has been thoroughly finished I shall think it undesirable to proceed to the succeeding chapter.

P. Before we commence reading the book, would you think it too much trouble to explain to me in a familiar manner the general aim and principles of the science which I am about to enter upon; for I find that communication *vivâ voce* is in general much clearer to me than the teaching of a book by means of dry definitions and axioms.

T. I do not object. I begin then by telling you that the Science of Mechanics is the Science of *Force.*

If we look about us we see that all things are in motion, or if not in motion they may be put in motion; a scientific mind at once inquires, What are the laws of this motion? there is a stone resting upon my hand, the stone is at rest, but if I take my hand away it falls, and it begins to fall gently, but soon its motion becomes more rapid; now I require to know the laws according to which this takes place. Again, I take this kitchen steelyard; here is a large weight hung upon this short arm, and here is a small weight hung upon this long one, and you see they just balance each other; now it becomes a question, what is the relation between the weights and the arms of the

* I take this opportunity of suggesting to all young students the great advantage of *writing* any proposition which they may desire to master, instead of merely *reading* it. The process of writing, though it may seem tedious, will, for the majority of students, prove the shortest and easiest method.

steelyard, in order that this may be the case. I could suggest a thousand similar questions, which it would puzzle you very much to attempt to answer. But here I must observe that I have just now proposed two questions to you of a very different kind; do you see the difference between them?

P. It struck me at the time that the questions seemed very different. For I think I could easily measure the length of the arms of the steelyard, and so ascertain the relation which they bear to the weights which hang upon them; but how to find out anything about the way in which the stone falls and how fast it goes, I confess this quite puzzles me.

T. You are quite right in your comparative view of the two questions. I should state the distinction in a shorter and more technical way by saying, that the question concerning the weights on the steelyard was a *Statical* question, that concerning the falling stone a *Dynamical*. Scientific men have seen it advisable to divide the science of Mechanics into two great sections, the first and simpler of which we call *Statics*, which treats of *bodies at rest*, the second and more complicated *Dynamics*, which treats of *bodies in motion*. But in connexion with this division of the subject let me give you a strict and formal definition of Force.

Any cause which produces or tends to produce motion in a body is called force.

P. You use the term *body:* I am not quite sure that I have a distinct conception of the meaning which you attach to the word.

T. All things in the world of the existence of which we become aware by means of our senses we call *matter;* thus, wood, lead, water, air, &c. all come under the general definition of *matter*, and any portion of matter we call a *body*. If the body be inconceivably small we call it a *particle*, and a *body* may be regarded as a *collection of*

particles; a particle may be called a body, but a body is not necessarily a particle.

Now you will at once see, that if there be a cause *tending* to produce motion in a body it may, or may not, actually produce motion; if it be counteracted by some other cause it will not produce motion, if it be not counteracted it will. For instance, here is a book lying upon the table, if the table were removed the book would fall; why?

P. In consequence of its weight, I suppose.

T. Yes; I might ask you what you mean by *weight;* but passing that by, its weight is a *force*, because it tends to move it; the table prevents it from moving; therefore the table exerts a force upon it. One of these forces tends downwards, the other upwards, and the result is that the book is at rest. Here then we have a case of a body acted upon by forces, and yet remaining at rest. To which branch of Mechanics then would this case belong?

P. To *Statics.*

T. Certainly. And if the table be suddenly removed the book falls; and then we should have a problem, which would belong to ——

P. —— *Dynamics.*

T. Yes; and you were quite right in supposing, that statical questions are the more easy of the two to treat. At present we shall be entirely employed with Statics, and when you are familiar with that branch it will then be time to proceed with Dynamics.

Now one of the first things which it will be necessary for you to lay hold upon clearly, is the manner in which a force is to be measured. Here is a weight of 1lb. lying on the floor; by means of this string which is attached to it and which I am, as you see, pulling directly upwards with my left hand, I am exerting a force upon the weight. Here is another weight of 2lbs., upon which I am, in like manner, exerting a force by means of a string which I

hold in my right hand. I have now lifted them both, the one with my left hand, the other with my right; what conclusion do you draw concerning the *force* which I am exerting by means of my right and my left hand, respectively?

P. One is twice as great as the other, since your right hand supports twice as great a weight as the left.

T. Certainly; and you have now told me the method by which we measure all forces in Statics; I will enunciate it for you thus:

A Statical force, or a force considered statically, is measured by the number of pounds weight which it will sustain.

So far as our purposes are concerned at present, it is not at all necessary to consider what a pound weight is: it is sufficient to observe, that certain standard pound weights exist, which are made very accurately, and preserved with great care, and these standard weights determine the pound weights throughout the country. If then we call a force which will lift a pound weight unity, or 1, then a force which will lift two such weights will be denoted by 2, and, generally, a force which will lift P pounds will be denoted by P.

P. I observe that in this book the author speaks of a force P, a force Q, and the like; I suppose that what you have just now said explains what he means.

T. Doubtless; we speak for shortness' sake of a force P, meaning by the expression *that force which would just lift* P *pounds;* and if you bear this in mind each symbol you meet with will convey to you a distinct meaning. When we come to the consideration of Dynamics we shall be obliged to enter more particularly into the question of measuring force.

P. I observe also that straight lines are spoken of as representing forces: this appears somewhat strange; at least it does not very readily occur to me that a force and a straight line can have many common properties.

T. The representation of a force by a straight line is extremely convenient, and by no means so arbitrary as you seem to imagine. For, let us analyse the conception of a straight line and that of a force, and we shall see that there are two things which are necessary to determine a straight line, and that the same two things are necessary to determine a force. Do you see what these are?

P. I am not sure that I understand clearly what you mean by the expression *determine* a straight line, and *determine* a force.

T. In reading Euclid you will remember that we find the phrase a *given* finite straight line; what do you mean by saying that a finite straight line is given?

P. I mean that I know its length and how it is situated.

T. Precisely so: in other words, you know its *magnitude,* and you know its *direction.* It would not be sufficient to know the magnitude without the direction, nor the direction without the magnitude; but if we know these two things we know all that it is possible to know concerning the finite straight line; and therefore the magnitude and the direction are said together to *determine* the straight line.

P. I believe I understand you: you would say that the position of the centre and the length of the radius *determine* a circle, and that two sides of a right-angled triangle determine the third. Am I right?

T. Certainly. And now let us consider what is necessary to *determine* a force. How did we say that the magnitude of a force was estimated?

P. By the number of pounds weight which it would lift.

T. Yes; but I shall not know all that may be known concerning a force by being told the number of pounds

which it can lift; for instance, you may have two locomotive engines of precisely the same power, but you could scarcely say with propriety that they were exerting the same force if one was dragging an up-train and the other a down-train: in one sense of course this would be true, but we should speak more accurately if we said, that the forces though the same in *magnitude* were opposite as to the *direction* in which they were exerted. Again, suppose you are felling a tree, and have a rope tied to the upper part of it in order to prevent it from falling upon your house, you would not think it sufficient to say how many men were to be employed in pulling the rope, but you would also tell them in which direction they were to pull; that is, you would give the *direction* as well as the *magnitude* of the force which you wish to be applied to the tree.

P. This seems to me quite clear: it is obvious that though two forces may be equal in magnitude, they may produce very opposite effects according to the direction in which they act, and therefore that to determine a force entirely we must know its *magnitude* and its *direction.* There is however one point which I should wish to have cleared up: how am I to define the *direction* of a force?

T. You may say that the direction of a force is that direction in which a particle on which the force acts would *begin to move,* if not prevented from moving by other causes.

Now let us collect the results of this discussion. We found that a finite straight line was determined by the knowledge of its magnitude and its direction; we have now concluded that a force is determined by its magnitude and its direction; does it not seem then to be a very natural course to represent a force, which we cannot see, by a straight line, which we can see, and which will probably assist us in carrying on our reasoning respecting force?

P. It seems so, when viewed in this light. But I fear that I shall be apt to confound the two, and perhaps

to forget sometimes that the lines which I have drawn represent forces.

T. You will of course be liable to such confusion, but a little attention will save you from errors; to assist you in this we will always take capital letters as P, Q, R... to represent forces, and small letters a, b, c,... to represent lines.

There is one more point to which I must call your attention, while we are upon the subject of representing forces by lines, or as it may be called representing forces *geometrically*. Suppose I take AB to represent a certain force which we will call P;

then I intend to assert that AB is P times as great as a certain line, an inch for instance; so that if a line of 1 inch be conventionally taken to represent 1 lb., a line of 2 inches will represent 2lbs., and so on; and the position of the line AB upon the paper shews me the direction of P, or (as it is sometimes called) *the line of P's action;* but at present there is a point not clearly defined, and that is, whether the force is tending to move a particle from A towards B, or from B towards A; to make this appear to the eye by a figure, it is usual to denote the direction of a force by an arrow-head, the force tending to move a particle in that direction in which the arrow appears to be moving.

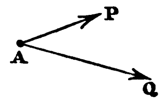
Thus if I wished to represent a force acting upon a particle A and tending to make it move from left to right across the paper, I should draw an arrow-head as in this figure.

And I might either take the line AB to represent the magnitude of the force, or (which is more common) I might at the point of the arrow put a letter or figure to indicate the magnitude of the force; thus I might represent as in the figure a force of P lbs. tending to move the particle A in the direction indicated by the arrow-head.

P. This seems to be very convenient, because it

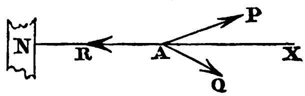
represents to the eye all that we can wish to know concerning the force. I suppose that the same method would enable us to represent several forces acting on a particle?

T. Certainly. A particle may be under divers influences; that is, it may be under the action of divers forces; and (as you have anticipated) these may be most conveniently represented by lines drawn in different directions, or by lines and arrow-heads. Thus, the particle *A* may be under the action of two forces *P* and *Q*, acting in different though not in exactly opposite directions, and we should represent this state of things by such a figure as I have just now drawn.



P. The particle *A*, if acted upon in the manner which you have represented cannot, I suppose, be at rest?

T. Surely not. If two equal forces act in exactly opposite directions upon a particle, it is clear that the particle will not move; but *two* forces, whether equal or unequal, cannot keep a particle at rest under any other conditions. The smallest number of forces which can keep a particle at rest, or (as we express it) *maintain equilibrium*, is *three.* Thus suppose we have two forces *P* and *Q* acting upon a particle *A*, and suppose that the particle would begin to move in the direction *AX*; produce *AX* backwards to *N*, and suppose *AN* to be a string made fast at *N*; then it is evident that the particle cannot move at all; and therefore it will be possible to find a force *R* which acting in the direction *AN* will counteract the combined effect of the forces *P* and *Q.* Thus it is clear that three forces may be found to keep a particle at rest, and also that if any two forces be given a third may always be found which shall counteract the effect of the other two.



P. It seems from what you have said that such a force

might be found; but I must confess that I have no notion how to set to work to find it.

T. I am not surprised: for this is in fact the very problem of Statics, and we may therefore now with propriety pass on to the consideration of the subject as it is treated in the book which I have put into your hands. I have only one preliminary remark to make; you will observe that the first two chapters are called *Experimental Mechanics*, and the fourth and fifth *Demonstrative Mechanics*. The fact is that the science of Statics is strictly a demonstrative science; in other words, it can be shewn, not only that the recognized laws of force are such as experiment indicates, but that they are necessarily true; indeed the great beauty of Mechanics as a mathematical science is that it possesses this demonstrative character; at the same time the subject presents so many difficulties to a young beginner, that I have thought it a convenient method to introduce the fundamental laws of Statics in the first instance as results of experiment, and afterwards to demonstrate the truth of the same laws mathematically. The advantage of this method is, that when you arrive at the demonstrative portion you will have a clear conception of what it is that you are required to prove; and in many cases to see clearly what it is that is to be done is a very considerable step towards actually doing it.

CHAPTER I.

EXPERIMENTAL MECHANICS. COMPOSITION AND RESOLUTION OF FORCES WHICH ACT AT ONE POINT.

1. THE simplest case of two forces acting at the same time upon the same particle, is that of two forces acting upon it in the same direction.

It is evident that a particle under the action of two forces cannot be at rest unless the two forces be exactly equal in magnitude and opposite in direction. Let us denote by P and Q two forces acting upon a particle in opposite directions, then for equilibrium we must have

$$P = Q \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

$$\text{or}\ \ P - Q = 0 \quad\ldots\ldots\ldots\ldots \quad (2).$$

2. If the forces P and Q be not equal, the greater will preponderate; and the particle will be under exactly the same circumstances as if it were acted upon by the excess of the greater over the smaller force. For instance, if a particle be acted upon by a force of 5lbs., tending to draw it from left to right across the leaf of this book, and by a force of 3lbs., tending to draw it from right to left, the particle will be under exactly the same circumstances as if it were acted upon by a force of 2lbs., tending to draw it from left to right. In this case the force of 2lbs. is said to be the *resultant* of the two opposite forces 5lbs. and 3lbs. More generally, if the two forces P and Q act in opposite directions, and P be the greater, $P - Q$ will be the *resultant* of P and Q, and will tend in the same direction as P. The resultant of these two forces may be denoted by R, and we shall then have

$$P - Q = R \quad\ldots\ldots\ldots\ldots\ldots \quad (3).$$

It is evident that if P act upon the particle and tend to draw it in one direction, and Q together with R tend to draw it in the opposite direction, the particle will be at rest.

If P and Q tend to draw the particle in the same direction, and we call R their resultant, we shall have in like manner

$$P + Q = R \;..............\; (4).$$

And generally, if a particle be acted upon by any number of forces P_1 P_2 P_3 ... tending to draw it in one direction, and by any number Q_1 Q_2 Q_3 ... tending to draw it in the opposite direction, and we call R the resultant, we shall have

$$P_1 + P_2 + P_3 + \ldots - Q_1 - Q_2 - Q_3 - \ldots = R \;......\; (5).$$

3. It is frequently convenient to use a symbol for a force which shall indicate not only the magnitude of the force, but also the direction in which it acts. Now in Trigonometry we make use of the signs + and − to indicate the directions in which straight lines are drawn, and very great advantage is derived therefrom. If we take a fixed point A, and draw a straight line AB of length a in a given direction, and then draw a straight line AB' of the same length (a) in the exactly opposite direction, we distinguish AB from AB' by calling $AB + a$ and $AB' - a$. And it is quite unnecessary to explain to any person who is acquainted with Trigonometry the remarkable simplicity and generality which is given to formulæ by this means. Suppose then we adopt a similar convention respecting forces; that is, if A be a particle acted upon by a force P tending to move it from A towards B, let us denote the force by $+ P$, and then we can denote by $- P$ an equal force tending to move the particle from A towards B'.

4. We can by this convention enunciate, in a very neat and simple form, a proposition which expresses the

rule for finding the resultant of any number of forces, acting upon a particle along the same straight line, but some tending to move it in one direction along the straight line, and others to move it in the exactly opposite direction; for we may say that

The resultant of any number of forces having the same line of action is the algebraical sum of the forces.

Or if we denote by P_1 P_2 P_3 ... any number of forces acting in the same line, and by R their resultant, and if P_1 P_2 P_3 ... represent positive or negative quantities as the case may be, we shall have

$$P_1 + P_2 + P_3 + \ldots = R \text{ .... (6).}$$

5. We may further make use of the convention concerning positive and negative forces to enunciate the general condition of equilibrium of any number of forces acting upon a particle and having the same line of action; for we may say, that under such circumstances *the particle will be at rest if the algebraical sum of the forces be zero.*

6. The term *resultant,* which we have used in this very simple case, is one of much more general application. Whatever forces may act on a particle and in whatever directions, they will be equivalent to one single force; for if they be not such as to keep the particle at rest, or produce equilibrium, the particle will begin to move in a certain direction, and it can be prevented from moving by a single force acting upon it in the exactly opposite direction; call the force which is just sufficient for this purpose R, then R is in equilibrium with the whole system of forces; but so it would be with a force R applied in the direction in which the particle would begin to move; consequently the whole system of forces is equivalent to one single force R acting in that direction in which the particle would begin to move. And R is therefore termed *the resultant of the system of forces.*

7. The general problem of Statics may be said to be, to find the resultant of any system of forces. The problem

is much simplified by supposing the lines of action of all the forces to lie in the same plane, which for clearness of conception we will suppose to be the plane of the paper, and to this limited case we shall confine ourselves. It will be found, however, that notwithstanding this limitation the results at which we shall arrive will be applicable to a very large class of interesting questions. Our first step must be to solve this problem: Given the magnitude and direction of two forces (*P* and *Q*) which act upon a particle, to determine their resultant (*R*). Or, which is the same thing, Given that three forces acting in the same plane upon a particle keep it at rest, to find the relations which subsist between the magnitudes and directions of the three forces.

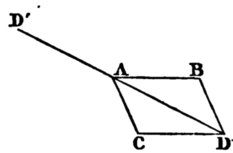
8. Now in considering the case in which a particle was acted upon by forces having the same line of action, we were able at once to deduce our results by reference to the simplest principles: but the problem with which we are now concerned does not admit of so easy a solution. There are two modes of solving it; we may either have recourse to experiment, or we may demonstrate it mathematically by means of certain axioms concerning force. In this treatise we shall adopt both methods; first we shall shew how to deduce the result experimentally, and we shall illustrate the result by some applications; by this means the reader will become familiar with certain new conceptions, which may at first seem strange to him; but he must not rest satisfied with this mode of treating the subject, for experiment can only render a law very probable, and results obtained by experiment however precious they may be when we can obtain them in no other way, are altogether of a different order of value from those which we can demonstrate to be necessarily true.

9. We might commence with some experiments for the purpose of determining the relations which the resultant of two forces given in magnitude and direction will bear to the forces themselves, but it will be more simple to enunciate the result at which mathematicians have arrived,

in the form of a theorem, and then shew how the theorem may be experimentally verified. This theorem is usually called the *Parallelogram of Forces* and may be enunciated as follows;

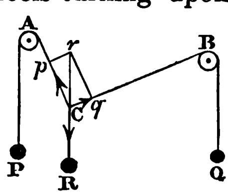
If two forces acting upon a particle be represented in magnitude and direction by two straight lines drawn from the particle, then the diagonal of the parallelogram described upon these two straight lines will represent the resultant in magnitude and also in direction.

Suppose, for instance, that A is the particle, and let it be acted upon by a force represented in magnitude and direction by AB, and by another force represented in magnitude and direction by AC; complete the parallelogram $ACDB$; then the resultant force will act in the direction AD and will be represented by AD in magnitude. So that if we produce AD backwards to D', and make AD' equal to AD, then AB, AC, AD', will represent three forces in equilibrium upon the particle A.



10. We shall now explain a method of proving by experiment the truth of this theorem.

Let A and B be two small brass wheels turning upon horizontal pivots inserted into a vertical board. The wheels must be carefully constructed so as to avoid friction as much as possible. Let P, Q, R be three weights attached to the extremities of three silk cords, the other extremities of which are all knotted together in one point C.



Now let the cords supporting two of the weights, as P and Q, be made to rest upon the wheels A and B, as in the figure, and let the weight R hang from the point C. Then it will not be difficult to arrange the system in such a manner that it shall rest as represented in the figure; we

have therefore the point C kept at rest by three forces acting in the direction of the three cords.

And these three forces will be measured by the three weights P, Q, R: for the effects of the wheels A and B is (neglecting any effect arising from friction,) merely to change the direction of the strings; that is, the force which must be applied at C in the direction AC in order to support the weight P must be a force whose measure is P. The point C is therefore kept at rest by three forces P, Q, R acting in the directions of the three cords which support the weights P, Q, and R respectively.

Along CA measure a line Cp containing as many inches as there are ounces in the weight P; and along CB a line Cq containing as many inches as there are ounces in Q; complete the parallelogram $Cprq$; and join Cr. Then it will be found that Cr is sensibly in the direction of RC produced, that is, vertical; and if Cr be measured, it will be found to contain as many inches as there are ounces in the weight R.

By these means, therefore, the Parallelogram of Forces can be proved, so far as such a proposition can be proved experimentally: the more accurately the experiments are made, and the more they are varied in their circumstances, so much the more certain will they render the truth of the proposition.

11. Let us now take a few illustrative examples.

Ex. 1. Suppose $P = 3$lbs., and $Q = 4$lbs., and the angle between them to be a right angle. Then it will be seen that the lines representing P, Q, and R form a right-angled triangle, and therefore by Euclid I. 47.

$$\therefore\ R^2 = P^2 + Q^2 = 9 + 16 = 25,$$

$$\therefore\ R = 5.$$

The angle which R makes with P is that whose cosine is $\frac{3}{5}$ or $.6$, which will be found to be $53^\circ\ 52'\ 11''$ nearly.

Ex. 2. Two forces act on a particle, the angle between their directions being 60°; to find the magnitude of their resultant.

Let P and Q represent the forces; R the resultant; then in the figure of page 17,

$$AB = P,\ BD = Q,\ AD = R,\ ABD = 180^\circ - BAC = 120^\circ;$$

$$\therefore\ R^2 = P^2 + Q^2 - 2PQ \cos 120^\circ$$
$$= P^2 + Q^2 + PQ, \text{ since } \cos 120^\circ = -\tfrac{1}{2}.$$

Ex. 3. Suppose in the preceding example $P = 2$ lbs., $Q = 3$lbs.

$$\therefore\ R^2 = 4 + 9 + 6 = 19;$$

$$\therefore\ R = \sqrt{19} = 4.3589 \text{ lbs.}$$

Ex. 4. To find the direction of R in the preceding example,

$$\sin BAD = \frac{BD}{AD} \sin ABD = \frac{3}{\sqrt{19}} \sin 120^\circ = \frac{3\sqrt{3}}{2\sqrt{19}},$$

$$\therefore\ BAD = 36^\circ\ 35'\ 12'',$$

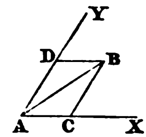
as will be found by help of a logarithmic table of sines.

12. When by means of the parallelogram of forces we find a single force which is equivalent to two others, we are said to *compound* those forces. And the parallelogram of forces may be described as the *rule for the composition of forces.*

Conversely we can by the same proposition find two forces which shall be equivalent to any one given force; when we do this, we are said to *resolve* the force in two others, or into two *components;* so that the parallelogram of forces is sometimes spoken of as the *rule for the composition and resolution of forces.*

If two component forces be given in magnitude and direction, it is a determinate problem to find the magnitude and direction of their resultant; but if the magnitude and direction of one force be given, it is not a determinate problem to find the two components of which it is the resultant, since there are an infinite number of pairs of forces, from the composition of which the given force may be conceived to have resulted. If, however, it be required to find two components in given directions from which a

given force shall result, the problem can be solved. Thus let AB represent the given force in magnitude and direction; AX, AY the given directions of the components. Draw BC, BD parallel to AY, AX respectively; then AC, AD will represent the magnitudes of the two components required.



Or, again, if one of the components be given in magnitude and direction, the other can be found. Thus, let AB be a given force as before, and let AC be one of its components; join BC, and complete the parallelogram $ADBC$, then AD will be the other component.

Ex. 1. A force of 6lbs. is the resultant of two forces, with the direction of which it makes respectively angles of 30° and 45°: required the magnitude of each component.

Referring to the preceding figure we shall have

$$AB = 6,\ BAC = 30^\circ,\ ABC = BAD = 45^\circ;$$

$$\therefore\ AC = 6 \times \frac{\sin 45^\circ}{\sin 75^\circ} = 4.3923 \text{ lbs.}$$

$$AD = BC = 6\frac{\sin 30^\circ}{\sin 75^\circ} = 3.1058 \text{ lbs.}$$

Ex. 2. Let the resultant be 10lbs., and let one of the components be 7lbs., and make with the resultant an angle of 60°, to find the other component.

Referring to the same figure, we have in the triangle ABC

$$AB = 10,\ AC = 7,\ BAC = 60^\circ, \text{ to find } BC;$$

$$\therefore\ BC^2 = 10^2 + 7^2 - 2 \times 10 \times 7 \cos 60^\circ$$

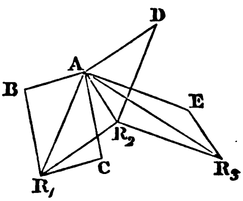
$$= 149 - 70, \text{ since } \cos 60^\circ = \tfrac{1}{2}$$

$$= 79;$$

$$\therefore\ BC = \sqrt{79} = 8.8882 \text{ lbs.}$$

13. It will further appear that the parallelogram of forces will enable us to find the resultant of any number of forces which act upon a single particle. For we have only to take two and find their resultant by the rule; then we can compound this resultant with the third; the next resultant with the fourth; and so on for any number.

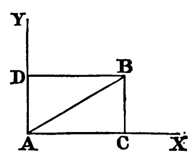
To represent this geometrically; let AB, AC, AD, AE, represent any forces acting on the particle A. Complete the parallelogram ABR_1C, and AR_1 will represent the resultant of AB and AC. Complete the parallelogram AR_1R_2D, and AR_2 will represent the resultant of AR_1 and AD, that is of AB, AC and AD. Lastly, complete the parallelogram AR_2R_3E, and AR_3 will represent the resultant of AR_2 and AE, that is, of AB, AC, AD and AE.



It is evident that this process applies equally well, whether the lines of action of the forces be all in the same plane or not.

We might, by means of the process here described, actually calculate the magnitude and direction of the resultant of any number of forces acting at one point; the method however would not be convenient, and we shall hereafter explain a process for finding the resultant, the same in principle, but much more easy of application.

14. We have observed, that a force may be resolved into two others in an infinite number of ways; there is, however, one mode of resolution which deserves particular attention, and that is the case in which a force is resolved into two components in directions perpendicular to each other. Let AB represent the given force; and AX, AY two directions at right angles to each other. From B let fall the perpendiculars BC, BD on AX, AY respectively; then since $ACBD$ is a parallelogram, AC, AD represent the components of the force in the directions AX and AY. Call the angle BAC θ; and let R be the original force, X and Y its two components. Then since

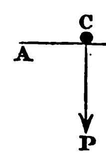


$$AC = AB \cos\theta, \quad \text{and} \quad AD = AB \sin\theta,$$

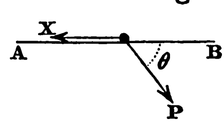
we have $X = R\cos\theta$, and $Y = R\sin\theta$.

Now the peculiarity of this case consists in this, that no force has any tendency to produce motion, in other words, it has no effect in the direction perpendicular to its own. This is evident from the very nature of force; but the truth of it is seen at once from such a consideration as this, that a body placed upon a horizontal table will be at rest, however smooth the table may be. The body is acted upon by a force, namely, its own weight, but this tends to draw it downwards, and the table being horizontal, will not admit of any motion downwards, consequently the body does not move at all. The pressure downwards is counteracted by the upward pressure of the table, and these two are exactly equal and opposite. The effect of the body's weight therefore is to produce a pressure upon the table, which is measured by the weight; but it has no tendency to produce motion upon the table, that is, there is no component in the direction parallel to the surface of the table, or perpendicular to the direction in which the weight acts.

The same thing may be seen as follows. Let C be a ball having a string attached to it, and let AB be a smooth plane having in it a rectilinear slit through which the string can pass. Then let a force P act at the extremity of the string; if the string be exactly perpendicular to the plane, it is obvious that there will be no motion, but if otherwise the ball will move along the plane in the direction of the slit. If the string be perpendicular to the plane, the force P will produce a pressure P upon the plane, and will have no component parallel to the plane or perpendicular to the string.



Suppose however that the direction of the string is not perpendicular to the plane; then motion may be prevented by a force acting along the plane; what will be the magnitude of this force? The preceding investigation will enable us to determine this. For let θ be the acute angle which the direction of the



string makes with the plane; then P may be resolved into two components, $P\cos\theta$ parallel to the plane, and $P\sin\theta$ perpendicular to it; let X be the force which would prevent the ball from moving, then it is evident that we must have

$$X = P\cos\theta.$$

And the component $P\sin\theta$ will produce a pressure upon the plane, having $P\sin\theta$ for its measure.

The component $P\cos\theta$ may be termed the *resolved part* of P parallel to the plane; and the preceding explanation will enable us to form a very distinct notion of what is meant by the *resolved part* of a force in any given direction; for it is measured by the force which is necessary to prevent the given force from producing motion in that direction. And we have this general rule of which it is impossiblet o overestimate the importance, *To find the resolved part of a force in any given direction, multiply the expression for the force by the cosine of the angle between the given direction and that of the given force.*

Suppose, for instance, that in the preceding figure, $P = 4$lbs., and $\theta = 60^0$; then the resolved part of P along the plane, that is, the force necessary to prevent motion along the plane $= 4 \times \cos 60^0 = 2$lbs. And the pressure upon the plane

$$= 4\sin 60 = 2\sqrt{3}\text{lbs.}$$

Again, suppose a horse is drawing a weight up a hill, and for distinctness let the weight be 1 ton, and let the inclination of the road to the horizon be 14^0; then the resolved part of the weight parallel to the road will be $1 \times \sin 4^0$, or 156.25lbs; and if we neglect the resistance to motion arising from the roughness of the road, this will be the measure of the effort which must be made by the horse in order just to drag the load. Practically this resistance may by no means be omitted, it is called *friction* and will be considered hereafter; but theoretically, that is, supposing the road be perfectly smooth, the horse would have to exert such a force as would just lift 156.25lbs.

And the resolved part of the weight perpendicular to, and therefore supported by, the plane will be

$$1 \text{ ton} \times \cos 4^{\circ}, \text{ or } 2234.5\text{lbs.}$$

15. We shall return to the subject of the Composition and Resolution of Forces, when we have demonstrated the Parallelogram of Forces independently of experiment.

CONVERSATION UPON THE PRECEDING CHAPTER.

T. I shall now be happy to explain any difficulties which may have occurred to you in reading this Chapter.

P. When the forces act in the same straight line there seems to be no difficulty whatever; it is a mere case of addition and subtraction. One point however occurs to me, concerning which I should wish to put a question. In what light am I to regard the principle of applying the signs + and − to indicate the directions of forces?

T. I think that the clearest manner of conceiving this is to regard the symbols + and − as marks indicating any qualities of bodies, or lines, or forces, which are exactly opposite to each other. When they are first introduced into mathematical reasoning, they are taken to represent the operations of addition and subtraction; and regarding them thus, $a - b$ means the number a when the number b has been subtracted from it; consequently if b be greater than a, the symbol $a - b$ has, properly speaking, no meaning, because a greater quantity cannot be subtracted from a less; but in Algebra we nevertheless often speak of negative quantities, which we denote by a symbol such as $-b$: what do we mean by such a symbol?

P. May we not say that it is a quantity which is *to be*

subtracted from any quantity greater than itself with which it may be connected?

T. Yes; you may regard it in that manner, and then a positive quantity would be considered as a quantity *to be added.* Now to add and subtract are exactly reverse operations, and a quantity which is to be added is exactly opposite in quality to one which is to be subtracted; and thus the symbols + and − when used as affecting monomial quantities, as $+a$ or $-b$, may be spoken of as symbols of *exactly opposite affections.* Observe then, that if instead of introducing + and − as symbols of addition and subtraction, we introduce them in the first instance as symbols of opposite qualities or affections, the use of them in elementary algebra will be no exception to this definition, but a particular instance of it. Various instances will at once occur to you, in which we are able to denote by such symbols opposite qualities which we have frequent occasion to express. Thus, let a man's property be a, and his debts b; then we should say that the man is actually worth $+a-b$; if a be greater than b the man is solvent, if $a=b$ he is worth nothing at all, and if a be less than b he is in debt; hence if we denote by p the man's actual property, supposing his debts paid, we have

$$p=+a-b,$$

and the state of the man's affairs depends upon the magnitude and sign of p.

Again, suppose that at a certain time a thermometer stands at exactly 0° or zero; suppose that it afterwards rises a degrees, and then sinks b degrees. If h be the height of the thermometer after this rise and subsequent fall, we shall have

$$h=+a-b;$$

if b be greater than a, b will be negative, and the thermometer will stand *below zero.* So that in publishing the state of the thermometer, it is not necessary to say that it was at a given time so much above zero, or so much below, pro-

vided that an algebraical sign be prefixed to the quantity which marks the height.

In like manner depth may be called negative altitude; and if the height of a body above the earth's surface were said to be – 20 feet, it would be understood that the body was in reality 20 feet below the surface of the earth.

It is precisely in the same manner that a *thrust* may be regarded as a negative *pull*, and that a force which tends from the earth's surface and is measured in magnitude by 4 lbs. might be spoken of as a weight of – 4 lbs.

And if you bear in mind this fundamental notion of + and – being marks which indicate opposition of some particular quality, you will easily see that if a force tending in one direction be affected by the sign +, a force tending in the opposite direction must be affected by the sign –, this being the only meaning which we attach to that sign.

P. This appears to me intelligible. With regard to the experimental proof of the Parallelogram of Forces described in this chapter, is there any difficulty in making the experiment?

T. None, except that which arises from the necessity of having the wheels *A* and *B* sufficiently carefully constructed to avoid the effects of friction. If you think it worth while, you can obtain the apparatus made ready to your hand*; you may also obtain a variety of other apparatus having the same purpose, namely, to exhibit the truth of the parallelogram of forces: but I do not wish that you should lay too much stress upon these experimental proofs; I would rather have you regard them as provisional, as a convenient mode of introducing you to new conceptions, but not as being intended ultimately to bear the weight of the proposition. When you have proceeded a little further you will perceive that the doctrines of Statics may be made to rest upon axioms peculiar to the subject, just as those of Geometry rest upon axioms pecu-

* All apparatus of this description may be obtained at Messrs Watkins and Hill, Charing Cross.

liar to that subject; and when the Parallelogram of Forces has been so established Statics will have to be regarded as a *demonstrative* science, not an *experimental* one, exactly as Geometry is *demonstrative* and not *experimental.* You might if you pleased, demonstrate the propositions of Euclid experimentally; you might by actual measurement prove, as far as such a process could prove, that the three angles of a triangle make up two right angles, or that the squares described upon the two sides of a right-angled triangle are equal to the square described upon the hypothenuse; but the doctrines of Geometry, rest, as you well know, upon quite different grounds, and though we have at present assigned an experimental basis to Mechanics, yet this is, as I have said, only provisional, and eventually I shall hope that you will regard the truths of Geometry and of Statics as resting upon similar foundations.

P. But will it not answer our purpose very well to confine ourselves to experiment as the basis of Mechanics?

T. Regarding Mechanics practically I should answer, Yes; but regarding Mechanics scientifically, No. If we used geometry only for practical purposes, it would be sufficient to be able to say that Euc. I. 47 was true as nearly as could be ascertained by experiment, and I have heard of persons who did not believe the truth of the proposition until they had tested it experimentally; but no experiment can shew that the proposition is accurately and mathematically true, and the position in which the human mind places itself when it affirms a proposition to be true is very different from, and much higher than, that which belongs to it when it has to depend for its convictions upon experiment.

P. Yet I have seen it stated that Mechanics must ultimately depend upon experiment.

T. I should rather say that Mechanics must depend upon experience than upon experiment; upon experience it must depend in this way, that we can have no notions

concerning force at all, prior to the experience which we have of the effects of force in the world about us; by consideration of the nature of force as we see it manifested in its effects we are led to certain axioms concerning force, and from these we are able without any direct experiment such as have been described in the preceding chapter to deduce the rules for the composition and resolution of forces, to construct in fact a complete system of Mechanics. Thus Mechanics may and ought to be regarded as a demonstrative science; but there are certain portions of it, as you will find, in which we are obliged to have recourse to experiment; for instance, the laws of friction are so deduced; but all such laws are to be considered not as absolutely true, but as sufficiently near to the truth for ordinary purposes.

P. I believe I understand you; I shall probably understand you better hereafter. Even with my present knowledge of Mechanics I find my mind much cleared respecting the action of force: for instance, I should have been disposed to say that the greater the steepness of a hill the greater would be the effort required to draw up a burden upon it, but I find now that strictly speaking I should speak of the *sine* of the hill's inclination.

T. We are apt to use phrases in common conversation which are deficient in scientific strictness; in the instance which you have adduced, the effort would vary directly as the steepness for very small inclinations, but if the inclination became considerable the law would be very different.

You cannot dwell too much upon the notion of resolving forces, to which you have been introduced in this chapter. It may be said to be the foundation of all mechanical science; and you will have made an important step, if by studying this chapter and working at the examples which follow you become entirely familiar with the conception.

EXAMINATION UPON CHAPTER I.

1. Define *force*, and explain how force is measured in Statics.

2. Distinguish between *Statics* and *Dynamics*.

3. Explain the method of representing forces by straight lines.

4. Enunciate the parallelogram of forces, and shew how it may be proved experimentally.

5. Two equal forces act upon a point, and the angle between their directions is 60°; find the magnitude and direction of the resultant.

6. A particle in the centre of an equilateral triangle is urged towards two of the angular points by forces each equal to 3 lbs.; find the force which must tend to the third angular point in order that the particle may be at rest.

7. A particle is acted upon by a horizontal force of 3 lbs. and a vertical force of 4 lbs.; find the direction and magnitude of the resultant force.

8. If two forces act at right angles to each other, they and their resultant are proportional to the sides of a right-angled triangle.

9. Three forces measured by 3, 4, and 5 pounds respectively keep a particle in equilibrium; determine the angles at which they act.

10. The resultant of two forces acting at right angles to each other is double the smaller of the two; find its direction.

11. The resolved part of a force in any direction is found by multiplying the expression for the force by the cosine of the angle between the direction of the force and the given direction.

12. A weight of 112 lbs. rests upon an inclined plane making an angle of 80^{0} with the horizon: what is the resolved part of the weight in the direction of the plane? and what the resolved part perpendicular to it?

13. Is it possible for three forces in the proportion of 3, 7 and 11, to be in equilibrium when acting upon a point.

14. Three forces act at a point, and their directions make angles of 120° with each other: shew that the three forces are equal.

15. The resultant of two forces which act at right angles to each other is 1 cwt.; and the direction of the resultant divides the right angle in two of 18° and 72° respectively: find the components in lbs.

16. If two forces P and Q acting upon a point have a resultant R, and θ be the angle between the directions P and Q, then

$$R^2 = P^2 + Q^2 + 2PQ\cos\theta.$$

17. Three forces of 2lbs. 7lbs. and 8lbs. are in equilibrium: find the angle between the directions of each two of the forces.

18. The square of the line representing the resultant of two forces is greater or less than the sum of the squares of the lines representing the components according as the angle between the components is acute or obtuse.

19. The magnitude of two forces being 25lbs. and 36lbs., and the angle between their directions 72°; find the magnitude and direction of the resultant.

20. The angle between two forces is 60°, and the resultant divides the angle in the ratio of 1 : 3; find the ratio of the components.

21. If P, Q, R be three forces which produce equilibrium upon a point, and if (PQ) denote the angle between the directions of P and Q, then

$$P : Q : R :: \sin(QR) : \sin(RP) : \sin(PQ).$$

22. Three forces acting at a point are in equilibrium, their directions making with each other the angles 60°, 135°, and 165° respectively; find the ratios of the forces.

23. If two forces acting on a point be in the ratio of 3 : 4; find the angle between them when the resultant is a mean proportional between them.

24. The resultant of two forces cannot be an arithmetical mean between them, if one of the forces be more than three times as great as the other.

25. If three forces be in equilibrium, any two must be together greater than the third.

CHAPTER II.

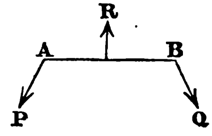
EXPERIMENTAL MECHANICS. THE PRINCIPLE OF THE LEVER.

1. In the preceding chapter we have treated of the Composition and Resolution of Forces which act all at one point, or of the conditions under which a single particle can remain at rest when two or more forces are acting upon it. We shall now take the case of a body of any magnitude, which is acted upon by various forces, and shall endeavour to discover, by experiment, the conditions which must be satisfied in order that a body of this kind may be in equilibrium. But the problem in this form would be too complicated; and we shall therefore take a simple case, from which afterwards it will not be difficult to pass to the more general.

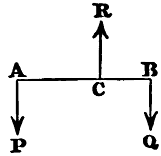
Our first simplification shall be this: instead of considering the conditions of equilibrium of a body of any form, we will take the simplest form of body possible, that is, a *straight rod*, which, however, we shall suppose to be so strong as not to bend under the action of any forces applied to it.

Our second simplification shall be as follows: instead of considering the case of any forces whatever acting upon this stiff or *rigid* rod, we will suppose it to be acted upon by only *three*, and we will suppose the lines of action of these forces to lie all in one plane.

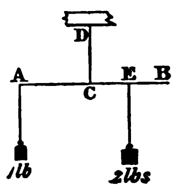
The system may then be represented as in the figure; AB is the rod; P, Q, R are three forces, whose directions lie in the plane of the paper, acting at three points A, B, and C respectively; the problem is to determine under what conditions the rod AB will be at rest.



There is yet one further simplification which we can make; and this consists in supposing the three forces P, Q, R to be parallel in direction, and that direction perpendicular to AB; and the system will then be represented by the figure; in which it will be observed, that we have represented one of the forces acting in a direction diametrically opposite to that of the other two, as must evidently be the case, since equilibrium could not subsist if the forces all tended in the same direction.



2. The problem is now in a form convenient for experiment; let a rod AB be suspended by a string, one extremity of which is attached to the middle point C of the rod, and the other to some fixed point of support D; now let any given weight, as a weight of 1 lb., for instance, be suspended by a string from the extremity A, and let some larger weight be attached to another string which by means of a ring, or otherwise, is capable of being suspended from any point of the arm BC; then it will be found that if this second weight be suspended from the extremity B, B will descend until the three strings and the rod are all vertical, and by making the point of suspension more and more near to C, we shall come at last to a point from which, if the larger weight be suspended, the rod if placed in a horizontal position will remain so.

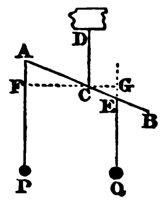


For instance, if the larger weight be 2lbs., it will be found that the point E, from which it must be suspended, is half-way between B and C; and in general, CE will be the same fraction of BC or AC that the smaller weight is of the larger; so that if we have two weights P and Q hanging by strings from two points A and E, and if $CA = p$, and $CE = q$, we shall have

$$P : Q :: q : p;$$
$$\text{or, } P . p = Q . q.$$

If we call the product of the weight P and the perpendicular upon its string from C the *moment* of P about C, then we may express the above relation by saying that *the moments of* P *and* Q *about* C *are equal.*

3. It will be found that if this condition between the two weights and the distances of their points of suspension from C be satisfied, the system will remain at rest when C is not horizontal; in this case if we drop the perpendiculars CF, CG from C upon the strings supporting P and Q respectively, we have by similar triangles



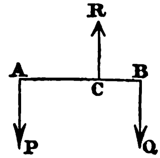
$$CF : CA :: CG : CE;$$
$$\therefore P \times CF = Q \times CG,$$

or the moments of P and Q about C are still equal.

4. In this experimental investigation, we have hitherto only spoken of the *two* weights suspended from A and E; the third force acting on the rod is the force exerted by the string CD; and what will be the magnitude of this force? It will evidently be measured by the weight which it supports, *i.e.* the two weights P and Q and the weight of the rod; if we omit the weight of the rod, then the force exerted by the string CD, or its *tension* (as it is usually called), will be upwards and will be equal to $P + Q$.

5. In considering the problem theoretically it is convenient to conceive of the rod AB as having no weight, in order that we may confine our minds to the two downward forces P and Q, and the upward tension of the string $P + Q$, but practically of course AB has weight, though it may be very small; it was in order to avoid the effect of this weight that we directed the rod to be suspended by its *middle* point C, for if this be done it is clear that the extremity A will have no more tendency to descend than the extremity B, and therefore the weights P and Q twist the rod precisely as they would if it had actually no weight.

6. The weights P and Q which we have been considering, exert forces upon the rod AB in the directions of the strings by which they are suspended; now it is easy to prove by observation that these strings are parallel, at least that they are *sensibly* parallel; also the string by which AB is supported exerts a force in its own direction, which is parallel to that of the other two strings; hence the preceding experimental investigation teaches us the conditions, under which three parallel forces can be in equilibrium when acting upon a straight rod. Let P, Q, R be the forces acting upon the rod AB, as in the figure, at the points A, B, and C respectively; then we must have



$$P + Q = R, \quad \ldots\ldots\ldots\ldots(1);$$
$$\text{and } P \times AC = Q \times BC \ldots\ldots(2).$$

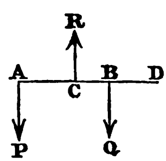
7. The latter of these two conditions may be put under another form; for if we write $R - Q$ instead of P in (2), we have,

$$(R - Q)\,AC = Q \times BC;$$
$$\text{or } R \times AC = Q\,(AC + BC) = Q \times AB;$$

i. e. the *moments* of R and Q about A are equal.

8. In like manner it may shewn, that the moments of P and R about B are equal.

9. And still more generally if we produce AB to any point D, we have



$$\begin{aligned} &P \times AD + Q \times BD \\ &= P \times (AC + CD) \\ &+ Q \times (CD - BC) \\ &= (P + Q)\,CD + P \times AC - Q \times BC \\ &= R \times CD \text{ by (1) and (2), (Art. 6).} \end{aligned}$$

i. e. the moment of R about D is equal to the sum of the

moments of P and Q about the same point. Now it is evident from inspection, that if we conceive AD to be a rod capable of twisting about one extremity D, and acted upon by three forces P, Q, and R, as in the figure, then P and Q tend to twist it one way and R tends to twist it in the other; hence we may say, that *there will be equilibrium, if the moment of the force which tends to twist the rod in one direction be equal to the moment of those tending to twist it in the other.*

10. Or we may express this condition still more conveniently, by introducing the use of the signs *plus* and *minus* to indicate the tendency of forces to twist the rod in opposite directions.

For if we call the moments of those forces which tend to twist the rod in one direction *positive*, and those which tend to twist it in the opposite direction *negative*, then it will be seen that the preceding conditions may be expressed by saying, that *the algebraical sum of the moments of the three forces about any point in the direction of the rod is zero.*

11. And it may be remarked that in like manner the condition (1), (Art 6), may be expressed by saying, that the *algebraical sum of the three forces is zero.*

Hence the complete statement of the conditions of equilibrium of three parallel forces acting upon a rod, will be as follows,

the algebraical sum of the forces $= 0$, ... (1)

the algebraical sum of the moments of the forces about any point $= 0$, ... (2).

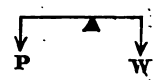
12. We shall defer the further generalization of these conditions, that is, their extension to any number of forces and to forces acting in any direction, to the Chapter in which we investigate the properties of moments of forces theoretically; the remainder of the present we will devote to the consideration of the problem of three forces acting upon a rod, exhibited under a somewhat different form.

13. Def. A rod capable of turning about a fixed point in its length is called a *lever;* and the fixed point is called the *fulcrum.*

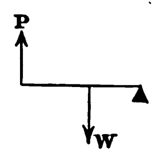
14. Def. If the lever be horizontal and a *weight* W be suspended from any point in its length, the lever may be sustained in a horizontal position by a certain force P acting at some other point, and tending vertically upwards or downwards according to circumstances. The force P which is required to maintain the equilibrium is called the *power.*

15. We may distinguish three classes of lever.

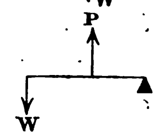
First. Suppose the *fulcrum* to lie between the *power* and the *weight.* This is called a *lever of the first kind.*



Secondly. Suppose the *weight* to act between the *fulcrum* and the *power.* This is called a *lever of the second kind.*



Thirdly. Suppose the *power* to act between the *weight* and the *fulcrum.* This is called a *lever of the third kind.*



16. Now it will be readily perceived, that although in all these cases we have spoken of only two forces, the *power* and the *weight,* as acting upon the lever, yet in reality there must be and are three forces. What is the third? It is supplied by the pressure upon the fulcrum. In the first case this pressure will be the sum of the power and the weight; in the second and third it will be their difference. Or if we call the pressure on the fulcrum R, we shall have

for the lever of the first kind $R = P + W$,

.... second... $R = W - P$,

.... third ... $R = P - W$,

But there will be this distinction between the pressure

on the fulcrum in the second case and the third, namely, that in the second (as in the first) the pressure is upwards, in the third it is a pressure downwards.

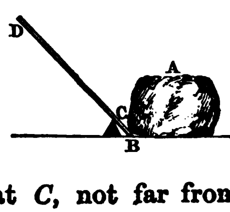
17. When we regard the lever thus, we see that all three cases are instances of the equilibrium of three parallel forces, and that the conditions of equilibrium will be those which have been already investigated. In each lever therefore we must have,

moment of the power about the fulcrum
= moment of the weight.

If then we suppose that we have a lever with a given fulcrum, and a given weight W suspended at a given point, we can at once determine the point at which a given power P must act in order to be in equilibrium with W; or, if the point of application (or as it is sometimes expressed *the length of P's arm*) be given, we can calculate P.

18. And the following conclusions will at once appear to be true.

(1) In the case of the lever of the *first* kind, we may make P as small as we please, if we increase the arm at which it acts in the same proportion that we diminish P. That is, a weight of any magnitude may be supported by any small force, if we give to this force a sufficient length of arm. Thus a weight of 100lbs. suspended at a distance of 1ft. from the fulcrum, may be sustained by a weight of 1lb. suspended at the other extremity of the lever, provided that that extremity be 100ft. from the fulcrum. And this points out to us the great advantage which may be gained in practice, by applying a force through the medium of a lever of this kind. Suppose for instance A to be a fragment of rock, a block of wood, or any other great weight which it is required to raise; let BD be a strong bar of iron or wood, and let one extremity B be inserted just under the weight A, and let the bar be made to rest against a support at C, not far from



the extremity B; then a comparatively small force applied at D will enable us to move A. This is the most ordinary application of the lever to common purposes, and probably every one is familiar with examples; in fact we make use of a lever of this kind every time that we rest the poker upon a bar to stir the fire; in this case the bar forms the fulcrum, the coals are the weight, the pressure of the hand on the poker is the power.

When by means of a lever we are enabled to make a certain force do an amount of work, which it could not do without the intervention of the lever, we are said to *gain a mechanical advantage*. It will be seen that mechanical advantage is not necessarily gained; thus, if in the preceding example the distance between the fulcrum and the point of contact between the lever and the mass A be greater than that between the fulcrum and the point of application of the force D, mechanical advantage will be *lost;* in other words, it would be easier to move the mass A by the direct application of the force than through the intervention of such a lever.

(2) In the case of the lever of the *second* kind, mechanical advantage is always gained. For the weight being between the power and the fulcrum, the arm of the power is necessarily greater than that of the weight, and therefore the power less than the weight.

We have an example of such a lever in the common nutcrackers; the pressure of the hand on the extremities of the long handles of the nutcrackers supplies the power, and the resistance of the nut the force which corresponds to the weight. Another example is that of the oar of a boat; the water forms a fulcrum, though an imperfect one, for the extremity of the oar, the hand at the other extremity supplies the power, and the result is a force greater than the power at the rowlock which is effective in moving the boat.

(3) In the case of the lever of the *third* kind, mechanical advantage is never gained. For the arm at which the power

acts is shorter than that at which the weight acts, and therefore the power must be greater than the weight. Hence it might be imagined that this species of lever could never be advantageously applied in practice, and of course if the gaining of power be the end to be attained it never can; there is however a most interesting case of the application of this kind of lever, in which the loss of mechanical advantage is far more than compensated by the gain of advantages of another kind. The case alluded to is that of the limbs of animals, or more particularly that of the human arm.

The figure represents the skeleton of the human arm; suppose the elbow *A* to be kept at rest, and the hand to exert a force either in lifting a weight, or in pulling, or pushing; then the tendency of the hand is to revolve about *A*, and *A* will be the fulcrum, while the force exerted by the hand will correspond to the weight. Where and how will the power be applied? The power is applied near the elbow by means of certain tendons or sinews, which are acted upon by the contraction of muscles situated in the higher part of the arm. Thus the point of application of the power is between the fulcrum and the weight, and the power acts at a mechanical disadvantage; but it will be easily seen from the nature of the case, that no other kind of lever could have been conveniently adopted, because the hand must of necessity be placed at the extremity of the limb; moreover, neatness of construction and agility of motion are incomparably more important in animal mechanism than the multiplication of strength, especially in the case of man whose natural strength must at best be small, and whose intellectual resources supply him with the means of increasing his power to an almost unlimited extent; the science of comparative anatomy, however, brings before us some curious instances of the power of the inferior animals being increased by advantageous mechanical arrangements.

A

19. We will now briefly recapitulate the results which we have arrived at in the case of the three levers respectively.

Lever of the first kind. Mechanical advantage may be either lost or gained.

Lever of the second kind. Mechanical advantage is always gained.

Lever of the third kind. Mechanical advantage is never gained.

And in all cases the principles of equilibrium is this, that the moment of the power about the fulcrum must be equal to the moment of the weight.

20. There are many other questions connected with the lever, or more generally with the theory of the *moments of forces,* which might be introduced in this place; we prefer, however, to reserve these until after we have treated of the Centre of Gravity, and have shewn how the principles already established by mere experiment may be placed upon a demonstrative basis.

CONVERSATION UPON THE PRECEDING CHAPTER.

P. I find the *moment of a force* to be here defined as the product of a force by a certain distance; I confess that I cannot form to myself any distinct conception of the product of force and distance. Can you help me to any clear thoughts upon this matter?

T. I am glad that you have called attention to the point, as it certainly requires explanation. Let us resume the proposition which gave rise to the introduction of the term *moment;* it was this, (page 32)

$$P : Q :: q : p.$$

Now in this proportion P and Q are forces, p and q are *lines:* P and Q are measured by the number of lbs. weight

which they will support, p and q are measured by the number of feet, inches, &c. which they contain; therefore the proportion may be written thus,

$$\text{number of lbs. in } P : \text{number of lbs. in } Q$$
$$:: \text{number of ft. in } q : \text{number of ft. in } p.$$

Thus the proportion becomes a relation between *numbers*, and we may transform it into this equation.

$$\text{number of lbs. in } P \times \text{number of ft. in } p$$
$$= \text{number of lbs. in } Q \times \text{number of ft. in } q,$$

and when this condition is satisfied P and Q are in equilibrium. The effect of a force then does not depend solely on its intensity, nor solely upon the arm at which it acts, but depends upon something which varies jointly as the two. Suppose I call this effect of a force acting at a certain arm its *moment;* then the question arises, how am I to measure this moment? Now in order to measure any quantity I must first fix upon some quantity which I can call unity or 1, and this I call the *unit of measure.* For example, if I wish to measure a distance, I call a foot unity, and then any length is measured by the number of *feet* it contains; and if I wish to measure a force, I call the force which will support 1 lb. unity, and then any force is measured by the number of *pounds* it will support. Precisely on this principle I must measure moment; and if I take for the unit of moment the moment of a force of 1 lb. acting at an arm of 1 ft., then the moment of any other force at any other arm will be measured by multiplying together the number of lbs. which measure the force, and the number of feet which measure the arm. In this way it is that the moment of a force is said to be the product of a force and a line; more strictly speaking, it is a quantity, whose numerical measure is the product of the numerical measure of the force and the numerical measure of the line. Does this explanation make the point clear?

P. I think so; and I conclude that when any quantities are said to be multiplied together it is their numerical

measures which are multiplied; so that when it is stated that the product of two lines makes an area, the thing intended is that the number of square feet in the area is the product of the numbers of linear feet in the two lines.

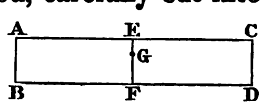
T. That is a correct view, and it is one of the utmost importance to be kept constantly before you in studying mechanics. Multiplication is a purely numerical operation, and applies to *abstract numbers* only, and not to the *concrete units* which those numbers may happen in any given instance to represent. Thus $3 \times 2 = 6$; but 3 shillings $\times$ 2 pence = nonsense; and hence you may see the absurdity of a problem which seems to have been invented for the express purpose of confusing the minds of young people upon an important fundamental principle; I mean the problem, "Multiply £19. 19*s.* 11$\frac{3}{4}$ by £19. 19*s.* 11$\frac{3}{4}$." This problem has simply no meaning. And this gives me the occasion of making another remark, namely, that quantities which represent forces and quantities which represent moments are of altogether different kinds and cannot be compared one with the other; you cannot say that a moment is twice as great as a force, or three times as great, any more than you can say that a mile is twice as long as an hour. And hence, if F represent a force and M represent a moment, an equation such as

$$F = M$$

would be absurd.

P. I believe I now understand this point. I am not sure whether I understand what is stated in Art 5, namely, that if the rod there spoken of be suspended by its middle point, the effect of the weights P and Q suspended from it will be the same as if the rod had no weight.

T. Suppose we take a bar of wood, carefully cut into the form of a rectangular board $ABDC$; draw EF bisecting AC and BD, and at G any point in EF insert a small horizontal pivot; let the bar be suspended upon the pivot, and let AC be made horizontal.



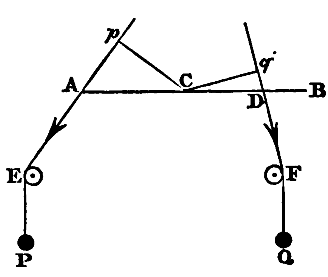
Then in whatever manner the weight of the portion *ABFE* tends to make that side of the bar descend by twisting about *G*, in the self-same manner must the portion *EFDC* tend to make the other side of the bar descend by twisting in the opposite direction about *G*. In fact, the two halves of the bar *ABFE*, *EFDC*, tend to descend equally, and therefore neither of them can descend, and the bar will remain in the horizontal position in which it was originally placed.

P. This need not be regarded then as matter of experiment?

T. Not necessarily; if it be admitted that the two halves of the bar tend equally to descend, it necessarily follows that neither of them can descend, or that the bar must remain horizontal, and if the bar has itself no tendency to turn about *G*, then it is evident that two forces which act upon it without twisting it must satisfy the same conditions as if they acted upon a bar of the same size and shape but having no weight.

P. Is not the experimental method of proof applicable to cases in which the forces acting are not perpendicular to the lever?

T. Yes; we might easily adapt the method as follows. Let *AB* be a rod moveable about a horizontal pivot through its middle point *C*; *E*, *F* two small brass wheels turning freely about their centres; and let two silk cords attached to the rod at two points *A* and *D*, pass over *E* and *F* and support two weights *P* and *Q*. Suppose the magnitudes of *P* and *Q* so adjusted that *AB* shall be horizontal; then if we suppose the directions of the cords *EA*, *FD* produced, and *Cp*, *Cq* drawn from *C* perpendicular to them, it will be found that



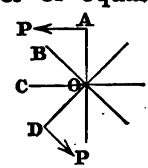
$$P \times Cp = Q \times Cq.$$

Now the two cords exert at A and D respectively forces which are measured by the weights P and Q; hence the preceding equation shews that in this case, as in that of forces acting perpendicularly to the lever, the condition of equilibrium is, that the moments of the forces about the fulcrum must be equal.

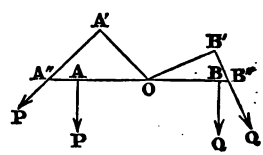
But this experimental demonstration is not worthy of much attention, partly because the simpler case will answer our purpose at present, so that we can defer the more general until we come to the demonstrative treatment of the subject, and partly because the truth of the more general case may be seen to follow without much difficulty from that of the more simple.

P. Will you explain this to me?

T. Let AO, BO, CO &c. be any number of equal spokes of a wheel, then it is evident that the effect of a force in turning the wheel will be precisely the same to whichever of the spokes it is applied. For instance, the force P applied at A perpendicular to OA will have the same effect as if it were applied at D perpendicular to OD.



Now take a lever AB, of which the fulcrum is O, and which is kept in equilibrium by two forces P and Q acting at A and B in the direction perpendicular to the lever. Through O draw OA' equal to OA, and OB' equal to OB, each making any angle with the lever; then from what has just now been said, it follows that P and Q acting at A' and B' perpendicularly to OA' and OB' will keep the bent lever $A'OB'$ at rest.



Again, let $A'P$ intersect OA produced in A'', and $B'Q$ intersect OB produced in B''; then if we conceive of the figure $A''A'OB'B''$ as one rigid board, capable of turning about O, and suppose the forces P and Q to act by means *of strings*, the directions of which coincide with $A'A''$ and

$B'B''$ respectively, it is evident that the effect will be the same at whatever point of $A'A''$ and $B'B''$ we suppose a tack to be driven through the strings so as to attach them to the board. Now if we suppose the string $A'A''P$ to be attached at the point A' we have the force P acting at the extremity of OA', and if we suppose it attached at A'' we have the force P acting obliquely at the extremity of OA''; hence the effect of P acting at A'' obliquely as in the figure is the same as that of P acting at A' perpendicularly to OA'. And hence we conclude that P and Q acting obliquely as in the figure at the extremities of a lever will produce equilibrium provided *their moments about the fulcrum are equal.*

Thus you will perceive that we can deduce the case of forces acting obliquely upon a lever from that of forces acting perpendicularly, and therefore if we wish to put the doctrine of the lever upon an experimental basis it is sufficient to make our experiment with weights suspended from a lever in a horizontal position.

EXAMINATION UPON CHAPTER II.

1. Define the *moment* of a force about a given point. A weight of 6 lbs. is suspended from one extremity of a horizontal rod; find the weight, which suspended from the middle point would produce the same moment about the other extremity.

2. Define a *lever;* and distinguish the different kinds of lever, giving examples of each.

3. Enunciate the condition of equilibrium of a straight horizontal lever, when a weight is suspended from each extremity; and explain how the condition may be investigated experimentally.

4. Enunciate in its most general form the principle of moments, as applied to the straight lever under the action of forces perpendicular to its length.

5. Shew how the case of forces acting obliquely upon a lever may be deduced from that of forces acting at right angles to the lever.

6. In what sense can a *moment* be properly spoken of as the *product of a force and a line*?

7. Two weights of 3 lbs. and 7 lbs. respectively, hang from the extremities of a lever 1 yard long; find the fulcrum.

8. A straight rod, 6 feet long, capable of moving in a vertical plane about one extremity has a weight of 10 lbs. suspended from its free extremity; find at what point an upward force of 35 lbs. must be applied so as to hold the rod in a horizontal position.

9. In the preceding example, what will be the pressure upon the fixed extremity?

10. What is meant by *mechanical advantage being lost or gained* by the intervention of a lever? Explain under what circumstances either the one result or the other takes place in the case of each kind of lever.

11. The longer arm of a lever of the first kind is 3 feet, and the shorter 7 inches; what force will be necessary to raise a weight of a ton.

12. How much would the force in the preceding example be increased by removing the fulcrum through 1 inch towards the weight?

13. Two weights, W and W', are suspended by strings from the extremities of a lever of length a; find the fulcrum.

14. If in the preceding example a weight w be added to W, what weight must be added to W' to maintain equilibrium?

15. A straight rod moveable in a vertical plane about a hinge at one extremity, is supported in a horizontal position by a vertical thread which is attached to it at a distance of 10 inches from the hinge; and the length of the rod is 27 inches. Supposing that the thread will support a weight of 4 oz. without breaking, find what weight may be suspended from the free extremity of the rod.

16. Two known weights, of P and Q lbs. respectively, balance upon a straight lever of the first kind; if p lbs. be added to P, the fulcrum must be shifted through a space a towards the extremity from which P hangs, in order to preserve equilibrium; and if q lbs. be added to Q, the fulcrum must be shifted through a space b towards the opposite extremity; find the length of the lever.

CHAPTER III.

ON THE CENTRE OF GRAVITY.

---

1. In every material body, or system of particles rigidly connected, there is a point which has this remarkable property, that if it be supported or fixed the body will remain at rest, whatever be the position of the body subject to the condition of that point being fixed. This point is called the *centre of gravity* of the body.

We shall be engaged in this chapter in proving the existence of the centre of gravity, in discussing some of its properties, and in determining its position in certain cases.

2. Let us take the simplest case, namely, that of two equal particles rigidly connected by a rod supposed to have no weight. Then it is evident that the middle point of the rod will be the centre of gravity; for the perpendiculars from this point upon two vertical lines drawn through the two particles will be equal, whatever be the position of the rod; therefore the moments of the weights of the particles will be equal, or the particles will be at rest.

3. Even if there be no rod joining the two particles, the middle point of the straight line joining them would be called their centre of gravity; for *if* this point were connected with the two particles, and the point were supported, the two particles *would* remain at rest in any position.

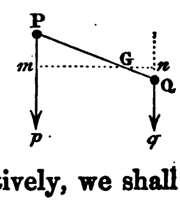
And generally, we may observe, that the centre of gravity of a body need not be a point within the body; but it may be, and frequently is, a point such that *if we conceive* the body to be rigidly connected with it the defi-

nition of the centre of gravity *would* be satisfied. For example, the centre of gravity of a hollow sphere is the centre of the sphere; for although that point has no physical connexion with the material sphere, yet if the centre be conceived of as rigidly connected with the sphere (by a rod without weight, for instance, coinciding with a diameter) it is evident that when the centre is supported the sphere will remain at rest in whatever position we place it; for the sphere being *symmetrical*, that is, of precisely similar size and shape, around the centre, when it is placed in one position there is no reason why it should change that position for another. Hence the centre of the sphere is called its centre of gravity, although there is no physical connexion between that point and the sphere itself. And so in other instances.

4. We may also observe, that a system of particles not rigidly connected are frequently spoken of as having a centre of gravity, as in the case of the two particles already discussed in Art. 3. By the centre of gravity in these cases we mean a point, which *if it were* rigidly connected with each of the particles *would satisfy* the definition given in Art. 1. In this sense we might speak of the centre of gravity of a heap of cannon-balls, of a quantity of water, of a piece of string.

5. Let us now consider what will be the position of the centre of gravity of two unequal particles.

Let P and Q be the two particles and let their weights be $p$lbs. and $q$lbs. respectively; draw a straight line from P to Q, and divide it in G in such manner that



$$PG : QG :: q : p;$$

then if from G we draw Gm, Gn perpendicular to the vertical lines Pp, Qq respectively, we shall have by similar triangles

$$PG : QG :: Gm : Gn,$$

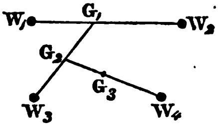
$$\text{and } \therefore\ Gm : Gn :: q : p.$$

Hence the *moments* of the two weights p and q about G will be equal, and therefore the two particles P and Q will balance about G; i. e. G will be the centre of gravity of P and Q.

6. We are now in a condition to prove the following general Theorem.

PROP. *Every system of particles and every material body has a centre of gravity.*

Let W_1, W_2, W_3, W_4,... be a system of particles, the weights of which are W_1, W_2, W_3, W_4,... respectively: suppose W_1, W_2 joined by a rigid rod without weight, and divide this rod in G_1, so that



$$W_1G_1 : W_2G_1 :: W_2 : W_1;$$

then, from what has gone before, G_1 will be the centre of gravity of W_1 W_2; that is, if G_1 be supported, W_1 and W_2 will balance in all positions about it, and the pressure upon the point of support will be $W_1 + W_2$.

Again, suppose G_1, W_3 joined by a rigid rod without weight, and divide it in G_2, so that

$$G_1G_2 : W_3G_2 :: W_3 : W_1 + W_2;$$

then, if we suppose the rod W_1W_2 to rest upon the rod G_1W_3, and G_2 to be supported, the pressure $W_1 + W_2$ at G_1 and W_3 at W_3 will balance about G_2. Hence the three bodies W_1, W_2, W_3, supposed rigidly connected, will balance in all positions about G_2.

Similarly, we may find a point G_3 in the line joining G_2 and W_4, about which $W_1W_2W_3W_4$ will balance in all positions, and so of any number of particles. Hence every system of particles has a centre of gravity.

And this proposition includes the case of all material bodies, since a body may always be conceived to be made up of an indefinite number of component particles.

Hence, every system, &c. Q.E.D.

7. If the centre of gravity of a system be supported, it is evident that the pressure upon the support will be precisely the same as if the whole system were compressed into a single particle having for its weight the sum of the weights of the particles of the system. This is sometimes expressed by saying, that *the statical effect of a system of particles is the same as if the system were collected at its centre of gravity.*

8. PROP. *Every material system has only one centre of gravity.*

For, suppose there are two, and let the system be so turned that the two centres of gravity lie in the same horizontal plane. Then the weights of the different particles of the system form a system of vertical forces, which must have a vertical resultant passing through each of the centres of gravity; otherwise the system could not balance about each of those points; hence the vertical resultant must pass through two points in the same horizontal plane, which is absurd. Hence every material system, &c. Q.E.D.

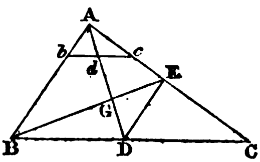
9. We shall now proceed to find the position of the centre of gravity in a few actual cases. The general determination of the position of the centre of gravity of a body of given form and magnitude we shall not be able to solve, but there are a few instances in which the problem does not present any difficulty.

10. *To find the centre of gravity of a physical right line, or of a uniform thin rod.*

The middle point will be the centre of gravity; for we may suppose the line or rod to be divided into pairs of equal weights equidistant from the middle point, and the middle point will be the centre of gravity of each pair, and therefore of the whole system, that is, of the line or rod itself.

11. *To find the centre of gravity of a plane triangle.*

Let ABC be the triangle; bisect BC in D, and join AD; draw any straight line bdc parallel to BC, and meeting AD in d; then by similar triangles, we have



$$bd : BD :: Ad : AD$$
$$:: cd : CD,$$
$$\text{or } bd : cd :: BD : CD;$$

but BC is bisected in D, therefore bc is bisected in d. Hence the line bc will balance about the point d in all positions; similarly, all lines in the triangle parallel to BC will balance about points in AD, and therefore the centre of gravity of the whole triangle must lie in AD.

In like manner, if we bisect AC in E, and join BE, the centre of gravity must be in BE; hence G, the intersection of AD and BE, is the centre of gravity of the triangle ABC.

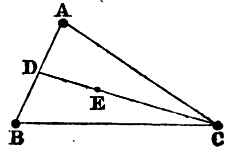
Join DE, which will be parallel to AB. (Euclid, VI. 2). Then the triangles ABG, DEG are similar;

$$\therefore AG : GD :: AB : DE$$
$$:: BC : DC$$
$$:: 2 : 1,$$
$$\text{or } AG = 2\,GD,$$
$$\text{and } \therefore AD = 3\,GD.$$

Hence, if we join an angle of a triangle with the bisection of the opposite side, the point which is two-thirds of the distance down this line from the angular point is the centre of gravity of the triangle.

12. *To find the centre of gravity of three equal bodies placed so as to form a triangle.*

Let A, B, C be the three bodies; join AB, BC, CA. Bisect AB in D, then D will be the centre of gravity of A and B, and we may suppose A and B to be collected at D. (Art. 7.) Join CD, and take DE equal to one-third of CD; then $CE = 2\,DE$,

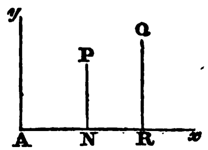


and therefore if we consider CD as a lever with fulcrum E, the two bodies A and B suspended from D will balance the body C suspended from C, and therefore E is the centre of gravity of the three bodies.

Cor. From this it appears that the centre of gravity of a plane triangle is the same as that of three equal bodies placed at its angular points.

13. We shall now shew how to find the centre of gravity of any number of particles in the same plane; but before doing so, we must shortly explain how the position of any number of particles may be most conveniently represented mathematically.

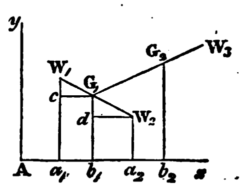
Let P be any point the position of which we wish to describe: take any point A, and through it draw two straight lines, Ax, Ay, at right angles to each other; from P draw PN perpendicular to one of these lines, as Ax: then it will be easily seen, that if the length of AN be given, and also the length of PN, the position of P will be entirely described. In like manner the position of any other point Q may be determined; and we may remark that this mode of assigning the positions of points is very general in modern mathematics, and it is usual to call AN, PN the *co-ordinates* of the point P, and to call Ax and Ay the *axes of co-ordinates*.



This being premised, let it be required

14. *To find the centre of gravity of any number of particles which lie in the same plane.*

Let W_1 W_2 W_3 ... be the weights of the particles; in the plane in which they lie take any two straight lines Ax, Ay at right angles to each other, as axes of co-ordinates. Draw W_1a_1, W_2a_2, ... perpendicular to Ax; and let $Aa_1 = x_1$, $W_1a_1 = y_1$, $Aa_2 = x_2$, $W_2a_2 = y_2$, &c., also let x, y be the co-



ordinates of the centre of gravity of the system; then it is evident that if we find x and y, we shall have solved the problem.

Join $W_1 W_2$, and let G_1 be the centre of gravity of W_1, W_2; from G_1 draw G_1b_1 perpendicular to Ax, and G_1c perpendicular to W_1a_1, also from W_2 draw W_2d perpendicular to G_1b_1; then, by the fundamental property of the centre of gravity, we have

$$W_1 \times W_1G_1 = W_2 \times W_2G_1;$$

but since the triangles W_1G_1c, G_1W_2d are similar, we have

$$\begin{aligned} W_1G_1 : W_2G_1 &:: G_1c : W_2d, \\ &:: a_1b_1 : a_2b_1, \\ &:: Ab_1 - x_1 : x_2 - Ab_1; \end{aligned}$$

$$\therefore\ W_1(Ab_1 - x_1) = W_2(x_2 - Ab_1),$$

$$\text{or } Ab_1 = \frac{W_1x_1 + W_2x_2}{W_1 + W_2}.$$

If we consider another particle W_3, we may, in searching for the centre of gravity of the three W_1, W_2, W_3, suppose the two former to be collected at their centre of gravity G_1; hence if G_2 be the centre of gravity of the three particles, and we draw G_2b_2 perpendicular to Ax, we have

$$Ab_2 = \frac{(W_1 + W_2)Ab_1 + W_3x_3}{(W_1 + W_2) + W_3};$$

and if we put for Ab_1 its value already found, we have

$$Ab_2 = \frac{W_1x_1 + W_2x_2 + W_3x_3}{W_1 + W_2 + W_3};$$

and so on for any number of particles. Hence, we shall have

$$x = \frac{W_1x_1 + W_2x_2 + W_3x_3 + \ldots}{W_1 + W_2 + W_3 + \ldots}.$$

And in exactly the same manner we should find that

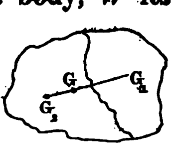
$$y = \frac{W_1 y_1 + W_2 y_2 + W_3 y_3 + \dots}{W_1 + W_2 + W_3 + \dots}.$$

It will be seen that these formulæ express this truth, that the centre of gravity of a system of particles is such, that the moment about any point A of the sum of their weights collected at the centre of gravity is equal to the sum of the moments of the weights.

15. If we have two bodies the centre of gravity of each of which is known, we can find the centre of gravity of the two, by considering each to be condensed into its centre of gravity, and then constructing for the centre of gravity of the two as we did for that of W_1 and W_2 in Art. 14. And the same remark applies to any number of bodies.

16. Also, when the centre of gravity of a heavy body is given, and also that of any portion of it, we can find the centre of gravity of the remainder.

For let G be the centre of gravity of the body, W its weight: G_1 the centre of gravity of the given portion, W_1 its weight. Join G_1G, and in that line produced, take G_2, such that



$$G_2G : G_1G :: W_1 : W - W_1.$$

Then G_1 will be the centre of gravity required.

17. The following general proposition concerning the centre of gravity, contains the property which is most important in a practical point of view.

Prop. *When a body is placed upon a horizontal plane, it will stand or fall according as the vertical line through the centre of gravity falls within or without the base.*

Suppose the vertical line GC through the centre of

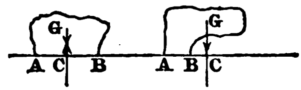


Fig. I. Fig. II.

gravity G to fall within the base, as in fig. I: then we may suppose the whole weight of the body to be a vertical pressure W acting in the line GC; this will be met by an equal and opposite pressure W from the plane on which the body is placed, and so equilibrium will be produced and the body will stand.

But suppose, as in fig. II, that the line GC falls without the base; then there is no pressure equal and opposite to W at C, and therefore W will produce a moment about B, (the nearest point in the base to C,) which will make the body twist about that point and fall.

18. Hence, we see that it is not necessary that the walls of a lofty building should be accurately vertical; this is in fact a condition which is very often not satisfied; and there are some very remarkable deviations from verticality, the leaning tower of Pisa for example.

19. We have used the term *base* in the preceding proposition to express the portion of the body which is in contact with the horizontal plane; if the body stand upon three or more points, then by joining these points, we shall form a triangle or polygon as the case may be, and this will be the space within which the vertical from the centre of gravity must fall.

20. It would seem from what has just been proved, as though a body would rest on a horizontal plane, when supported by a single point, provided that it be so placed that the centre of gravity is in the vertical line passing through that point, which in this case forms the base. And in fact a body so situated would be, mathematically speaking, in a position of equilibrium, though practically the equilibrium would not subsist; this kind of equilibrium and that which is practically possible are distinguished by the names of *unstable* and *stable*. Thus an egg will rest

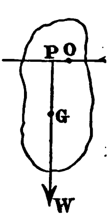
upon its side in a position of *stable* equilibrium, but the position of equilibrium corresponding to the vertical position of its axis is *unstable*. So likewise there is a mathematical position of equilibrium for a needle resting on its point, or a pyramid or cone upon its apex, though such positions are obviously unstable.

The distinction between *stable* and *unstable* equilibrium may be enunciated generally thus: Suppose a body or a system of particles to be in equilibrium under the action of any forces; let the system be arbitrarily displaced very slightly from the position of equilibrium, then if the forces be such that they tend to bring the system back to its position of equilibrium, the position is *stable*, but if they tend to move the system still further from the position of equilibrium it is *unstable*.

21. We shall conclude this Chapter with a property of the centre of gravity nearly analogous to that of Art. 17.

PROP. *When a heavy body is suspended from a point about which it can turn freely, it will rest with its centre of gravity in the vertical line passing through the point of suspension.*

For let O be the point of suspension, G the centre of gravity, and suppose that G is not in the vertical line through O; draw OP perpendicular to the vertical through G, that is, to the direction in which the weight of the body W acts. Then the force W will produce a moment $W.OP$ about O as a fulcrum, and there being nothing to counteract the effect of this moment, equilibrium cannot subsist.



Hence G must be in the vertical line through O, in which case the weight W produces only a pressure on the point O, which is supposed immoveable.

CONVERSATION UPON THE PRECEDING CHAPTER.

T. I think that this will be a convenient opportunity for me to say something to you concerning *gravity* or *weight.* On a former occasion (p. 6.) when you spoke of a body's *weight* being the cause of its fall if unsupported, I said that I might ask you the meaning of *weight;* but I thought it better to defer a discussion which might have puzzled you, until you had become more familiar with the conception of *force.* Let me now therefore ask that question,—what do you mean by *weight?*

P. I have no very distinct thoughts on the subject: I find that bodies require, some more, some less, exertion to lift them from the ground, and I call the resistance they make to being lifted their weight. But I know that this is a very insufficient account of the matter, and shall be glad to be enlightened.

T. When we speak of explaining natural phenomena, the fall of a stone for instance, all that we can pretend to do is this, to link together phenomena which to an uneducated mind appear quite unconnected, and to trace them to one common law. So that Science does not at all take away the necessity of belief in One who governs all things, it only reveals to us some of the laws according to which He works. Now one of the most general and most wonderful, though most simple, of the laws to which modern science has conducted us is this, that *every particle of matter attracts every other particle of matter towards itself.* Thus let *A* and *B* be any two particles; then by their very nature *A* tends to draw *B* towards itself, and *B* in like manner tends to draw *A* towards itself. No better illustration of this can be given than the phenomenon which we are now discussing, namely the fall of bodies; for if *A* be the earth and *B* a body raised from its surface, this becomes the case of a

A. *B.*

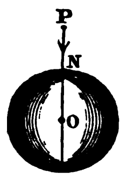
falling body. But there is an instance which being less familiar, will strike you more. Captain Basil Hall mentions, and it has been noticed by others, that when a fleet of ships are lying becalmed, the mutual attraction of bodies manifests itself in a very serious manner, for in consequence of their mutual attraction, there is a continual tendency in the ships to approximate to each other; and if two ships should be brought into contact, a slight but constant attrition is produced, imperceptible perhaps to the eye, but, in consequence of the enormous mass of the vessels, quite sufficient to perform a work of great damage. Captain Hall mentions that during the becalming of a fleet, it is necessary each morning to tow the heads of the vessels round, and so neutralize the effect which has been produced during the night.

P. Is this attraction of the same kind as that of the magnet?

T. They are alike in some respects; but there is this important distinction, that whereas the magnet attracts only certain substances, the attraction of matter, or as we call it, the *attraction of Gravitation* exists between *any* two bodies; the magnet will attract iron and will not attract wood; there is no such distinction between different substances in the case of gravitation.

In consequence of this law of gravitation every particle of the Earth's mass attracts every particle of a body at its surface*; and if we suppose (which is very nearly but not quite true) that the Earth is a *sphere*, then the *resultant* of the attractions of the particles of which it is composed upon a particle at the outside of it will be a force tending

* One of the most startling features of the attraction of gravitation is perhaps this, that the attraction of one particle upon another is in no way affected by the interposition between them of any mass of matter however large; thus the matter lying upon the other side of the earth, at our Antipodes, acts as powerfully upon a body on the side of the earth upon which we ourselves stand, as if the whole intervening mass of the earth were removed.



towards the centre of the sphere. Let, for instance, O be the centre of the Earth, P a particle anywhere outside the Earth; then every particle of the Earth's mass tends to draw P towards itself, and since these particles are all symmetrically arranged round the line OP, it is evident that the resultant of all their attractions must be a force in the direction PO. And *this resultant force is that which constitutes the weight of the particle* P.

P. When I speak of a body naturally falling *downwards*, therefore, I am expressing that it falls *towards the Earth's centre.*

T. Undoubtedly; except with reference to the Earth and its centre, *upwards* and *downwards* have no meaning; what we call *downwards* is, in fact, the same direction as that which the people of New Zealand call *upwards;* and if you were placed out of the reach of the Earth's sensible attraction, you would not be able to speak of *up* and *down* at all.

Now if in the last figure P were a stone which was let fall, it would of course fall in the direction of PO, or the direction of gravity, and this direction we will call the *vertical* direction; and the plane perpendicular to PO, at the point N, where the stone strikes the Earth's surface, we will call the *horizontal plane* at that point. It is clear that the vertical directions at two different points of the Earth's surface cannot be the same, that is, they cannot be parallel because they meet in O; but O is at a great distance from the surface, nearly 4000 miles, and, therefore, if we take two points on the Earth's surface at no great distance from each other, the vertical directions at those two points will be *nearly parallel.* For example, take two places a quarter of a mile apart; the circular measure of the angle between the vertical directions at those points will

$$= \frac{1}{4} \times \frac{1}{4000} = \frac{1}{16000}.$$

This is a very small angle, amounting to only a few seconds; hence you will see, that even at the distance of a quarter of a mile from each other the directions of gravity at two places may be taken to be parallel. In all problems, therefore, concerning heavy bodies, we treat of gravity as a *force which acts in parallel lines.*

P. You said that the supposition of the Earth being a sphere was very nearly, but *not quite,* true; what difference will the error in that supposition make in the result at which you have arrived?

T. None with which we need concern ourselves: I will, however, state to you what is the accurate truth.

The Earth instead of being spherical, as we have supposed, is what is called a spheroid; that is, it is slightly flattened at the poles, and if we were to take a section of it by a plane passing through its centre and its poles, it would be an ellipse of which the axes would be nearly equal*. The earth being of this form we cannot conclude that the force of gravity must at each point tend towards its centre†; we can, however, describe very simply the exact direction of gravity at any place upon the earth's surface; it is found that *the direction of gravity is the straight line perpendicular to the surface of still water at the given place;* this is a result which may be verified by experiment with an extreme degree of precision, and which also agrees with the results of mathematical investigation. Instead, therefore, of the definitions of *vertical* and *horizontal,* which I gave you just now, we ought, more properly, to speak of the *vertical at any place* as the straight line perpendicular to the surface of still water; and the plane perpendicular to it, that is, the surface of the water

* The *polar* diameter is 7899 miles, the *equatorial* 7925.

† There is another slight cause of deviation not considered here, namely, the rotation of the earth about its axis. The discussion of this belongs to Dynamics, but the effect is extremely small, and may be neglected.

itself as the *horizontal plane*. For all common purposes, however, we may regard bodies as tending to fall towards the earth's centre; and even if we take the more accurate definition of the direction of gravity, our former conclusion will be true, namely, that gravity may be considered as a force which acts in parallel lines.

As we are now speaking of parallel forces, I should wish you to notice, that in the investigation of Art. 14, the only assumption throughout is, that the forces corresponding to the directions of the weights W_1 W_2 ... are all parallel, and the result will therefore be true not only of weights but of *any system of parallel forces*. In fact, suppose we have this problem:

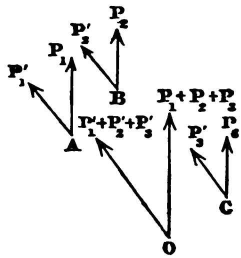
A system of parallel forces P_1, P_2, P_3, ... act at points whose *co-ordinates* (Art. 13) are x_1y_1, x_2y_2, x_3y_3 ...; required to find the co-ordinates of the point at which the resultant may be supposed to act. Then the investigation of Art. 14 will give us, *mutatis mutandis*, for the co-ordinates of the point required,

$$x = \frac{P_1x_1 + P_2x_2 + P_3x_3 + \dots}{P_1 + P_2 + P_3 + \dots},$$

$$y = \frac{P_1y_1 + P_2y_2 + P_3y_3 + \dots}{P_1 + P_2 + P_3 + \dots}.$$

Now there is a remark to be made concerning this point which is important: you will observe, that the *direction* of the forces $P_1, P_2, P_3, \dots$ does not occur in the result, which would therefore have been the same, whatever had been the directions of the forces, provided only that they were parallel among themselves. Hence, if we conceive of a number of different parallel systems, in which the forces have the same *magnitude* and the same *points of application*, but the *directions* different, then the resultants will have one common point, viz. that determined by the preceding formulæ, and this point is therefore called the *centre of parallel forces*.

To make my meaning clear, let A, B, C be three points at which the parallel forces $P_1P_2P_3$ act, and let O be the point in the direction of the resultant determined by the preceding formulæ; then if we suppose the directions of the forces changed so as to become AP_1', BP_2', CP_3', the direction of the resultant will be changed in like manner, but the point O *will still be a point in its direction.* Now suppose that we have a system of parallel forces acting at different points of a rigid body, and suppose the centre of the parallel forces is fixed, what conclusion do you draw?



P. That the body will be at rest.

T. Yes; and not only so, but that it would remain at rest however the directions of the forces were changed, provided that they remained parallel among themselves. Now you may regard the *centre of gravity* as a particular case of the *centre of parallel forces;* the weights of the particles of the body constitute a system of parallel forces; and you will see, from what has just been said, why it is, that when the centre of gravity is fixed the body will remain at rest in any position.

P. I believe I understand what you have been saying, but it seems to me to require some consideration.

T. It is, perhaps, rather hard for you to understand; but you will, I think, find advantage from considering the centre of gravity in the manner in which I have been representing it to you, namely, as a particular case of the centre of parallel forces. Have you any further question to ask?

P. When the centre of gravity of a triangle was investigated in Art. 11, no account was taken of the *thickness* of the triangle; and if it has no thickness, how can it have any gravity?

T. For simplicity's sake nothing was said concerning the thickness; but instead of speaking of finding the centre of gravity of a plane triangle, it would be more correct to speak of the centre of gravity of a body bounded by two plane triangles the surfaces of which are parallel; the centre of gravity of such a body as this manifestly lies in the plane which is parallel to the two bounding triangles and equidistant from them, and the place of the centre of gravity in that *plane triangle* will be found by the rule of Art. 11.

P. I suppose that it is upon a similar principle that the thin rod in Art. 10, is called a *physical* right line.

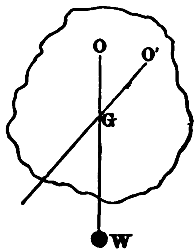
T. Yes: a right line as defined by Euclid has no breadth or thickness, and therefore can of course have no weight; therefore we cannot properly speak of finding the centre of gravity of a right line, that is, of a *geometrical* right line; but if you conceive of a rod of matter the transverse section, or thickness, of which is extremely small, as a fine wire, a thread, or the like, you may treat it as though it were actually a straight line, and if its thickness were indefinitely small, it would be called a *physical* line. *Geometrical* and *physical* are two terms which you will often find in this manner opposed to each other.

P. I should think it would not be difficult to illustrate the proposition in which the centre of gravity of a triangle is found, experimentally.

T. Nothing is more easy. Take a piece of cardboard in the form of a triangle; bisect two of the sides as accurately as you can, and join the points of intersection with the opposite angular points, then you will find that the card will balance upon the point of a pin slightly inserted at the centre of gravity so determined.

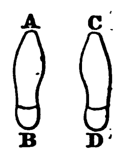
I would take this opportunity of remarking, that the position of the centre of gravity of any heavy body bounded by two parallel planes may be easily determined practically.

For this purpose let the body be suspended from a peg at any point O, in such a manner that it can easily turn about O; then by Art. 21 the centre of gravity will be somewhere in the vertical line through O: if, therefore, from O we suspend a plumbline, that is, a fine thread carrying at its extremity the weight W, and draw a fine line upon the bounding surface to mark the line of contact of the surface and the plumbline, the centre of gravity will be somewhere in this line. Again, let the body be suspended from a peg inserted at any other point O', and let a second line be traced by means of the plumbline, as before described. Then the point of intersection (G) of the two lines which we have traced will be the centre of gravity of the body. You would find some advantage, perhaps, from applying this experimental process to several cases, making use of pieces of cardboard of various irregular shapes.



P. Of course the conditions of equilibrium investigated in Art. 17 are those which apply to ourselves, are the conditions (I mean) of a man being able to stand upon the ground without falling.

T. Certainly: let AB, CD be the soles of a person's shoes; join AC, BD, then the vertical line through the person's centre of gravity must fall somewhere within the space $ABDC$. This space may be increased by separating the feet, and the man's *stability* is correspondingly increased. If a person raise one foot from the ground, then his *base* is reduced to the sole of the other foot and cannot be increased: his stability is therefore much diminished, and if he should lose his balance, he must either put the other foot down, or change his position by a *hop*, so as to bring the sole of his foot again below his centre of gravity.



Men, and indeed all animals, acquire the habit of instinctively shifting their position so as to satisfy the condition of equilibrium; thus, if a man walking upon a narrow plank feels himself in danger of falling upon one side, he throws out the opposite arm; a woman carrying a child leans backward; a man carrying a burden on his back leans forwards; in walking up a hill we lean forwards, in walking down we lean backwards; you will see at once the reason of these and the like actions.

In like manner you will perceive, that a person rising from a chair, must either press the body forward to bring the centre of gravity over the feet, or else put the feet backwards under the chair to produce the same effect. And in walking, the body requires to be thrown slightly forward, and the foot is advanced so as to support the centre of gravity.

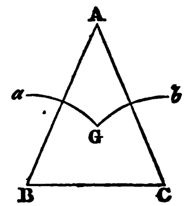
P. This I think I understand: has not the pole which tight-rope dancers carry in their hands something to do with the present subject?

T. Yes: and it is a very good illustration. When we speak of the centre of gravity of a man carrying a burden, we intend of course to speak of the centre of gravity of the two, considered as forming one body; and so the centre of gravity of the rope-dancer carrying a heavy pole, is the centre of gravity of the dancer and pole together. Now in consequence of this the rope-dancer may be said in a certain sense to carry his centre of gravity in his own hands, and he can shift its position, so as to keep it constantly within the narrow limits required, with much greater ease than he could if unassisted by the pole.

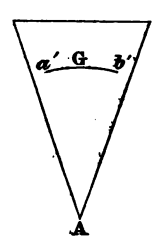
There is one other point concerning the centre of gravity which I think I can make intelligible to you, and which is worthy of your attention.

We have seen from the nature of the centre of gravity, that it is a point at which we may regard the whole weight

of a body as collected; now the tendency of weight is of course always to make a body *descend* not *ascend;* hence the tendency of the centre of gravity is always to descend, and if in any given case we can satisfy ourselves that the centre of gravity of a body is in the lowest position possible, we may be assured that it will not rise from that position, that is, the body will be in a position of equilibrium. This is sometimes expressed by saying that the centre of gravity seeks the lowest position. It cannot however be asserted that a body is in a position of equilibrium *only* when its centre of gravity is in the *lowest* position possible, for it will be found that (mathematically speaking) it is in a position of equilibrium when the centre of gravity is in the *highest* position possible. This I can easily illustrate. Let ABC be a heavy cone standing upon its base, G its centre of gravity: then if we make the cone to revolve about any point B in its base, the centre of gravity will trace out the circular arc Ga; if we make it revolve about C, it will trace out the circular arc Gb; and so on. In all cases G will rise, hence G is in its lowest position, and the cone will be at rest.



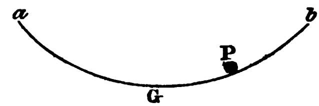
But suppose we place the cone upon its apex as in the figure; then if the axis is accurately vertical, the centre of gravity is vertically over the base as explained in Art. 20, and *theoretically* there will be equilibrium. Now if in this position we make the cone revolve about A, G will trace out a circular arc $a'Gb'$, of which G is manifestly the highest point: hence in this case the centre of gravity is in the *highest position possible,* and therefore what was just now stated is true, namely, that for such a position of the centre of gravity there may be equilibrium. But, as explained in Art. 20, the equilibrium in the first case is *stable,* in the second *unstable.*



P. That there should be equilibrium when the centre of gravity is in the lowest position possible seems almost obvious, but the other case does not seem so plain.

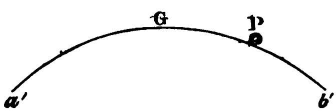
T. I think you will gain some light by considering the matter thus. In the first case the equilibrium may be compared to that of a *particle* which is placed upon a curve of this shape.

If we place a particle *P* upon such a curve what will be the result?



P. It will run down to *G.*

T. Yes; and it will remain there permanently, if placed there in the first instance. Now the second case is that of a particle *P* placed upon a curve such as $a'Gb'$: what will be the result?



P. It will run off the curve.

T. Certainly; except in one particular case, namely, when it is placed at the *highest point G.* In this case it will have no tendency to run towards a' rather than towards b', and will therefore remain at rest; but the least disturbance will make it run down the curve. Hence its equilibrium is unstable, or only theoretically possible; and a body may be in equilibrium when its centre of gravity is in the highest position possible, upon the same principle that the particle *P* can rest at the highest point of the curve $a'Gb'$.

EXAMINATION UPON CHAPTER III.

1. Define the centre of gravity of a body, or of a system of material particles.

2. Find the centre of gravity of two unequal particles.

3. Prove that every system of particles has one centre of gravity, and only one.

4. Explain and illustrate the statement that *the statical effect of a system of particles is the same as if the system were collected at its centre of gravity.*

5. Find the centre of gravity of a physical right line.

6. A straight wire 3 feet long is composed of two pieces of 2 feet and 1 foot respectively. The former is composed of matter which weighs 1 oz. per foot, and the second of matter which weighs $2\frac{1}{2}$ oz. per yard; find the centre of gravity of the whole wire.

7. Find the centre of gravity of a triangle.

8. Find the centre of gravity of three equal bodies placed so as to form a triangle.

9. Find the centre of gravity of any number of particles which lie in the same plane.

10. The centre of gravity of a body being given, and also that of a given portion of it, shew how to find that of the remainder.

11. An equilateral triangle is divided into two parts by a straight line which bisects two of the sides; find the centre of gravity of the quadrilateral portion.

12. When a body is placed upon a horizontal plane, it will stand or fall according as the vertical line through the centre of gravity falls within or without the base.

13. Distinguish between *stable* and *unstable* equilibrium.

14. When a heavy body is suspended from a point about which it can turn freely, it will rest with its centre of gravity in the vertical line passing through the point of suspension.

15. Explain what is meant by the *centre of parallel forces.*

16. Shew how to find the centre of gravity of a quadrilateral figure.

17. A body cannot be in stable equilibrium upon a horizontal plane if it rests on less than three supports, the supports being supposed to terminate in points.

18. Two unequal physical lines cross each other, and are attached at the point of their intersection: find their centre of gravity.

19. Find the locus of the centres of gravity of all right-angled triangles which can be described upon a given base.

20. If the sides of a triangle ABC be bisected in the points D, E, F; then the centre of the circle inscribed in the triangle DEF will be the centre of gravity of the perimeter of ABC.

21. A given number of weights (n), which are in geometrical progression, are placed at equal distances along a straight line: find their centre of gravity.

22. How may the centre of gravity of a plane figure be found experimentally?

23. Of all triangles upon the same base and having the same vertical angle, the isosceles is that of which the centre of gravity is nearest to the base.

24. Two rods of the same thickness, one of which is twice as long as the other, are attached by two of their extremities so as to be at right angles to each other. Find at what angle either of them will be inclined to the vertical, when they are suspended by a string or tack at the right angle.

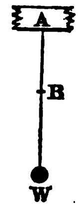
CHAPTER IV.

DEMONSTRATIVE MECHANICS. PARALLELOGRAM OF FORCES. EQUILIBRIUM OF A PARTICLE.

1. In a preceding chapter we explained fully the meaning of the proposition called the "Parallelogram of Forces," and we deduced its truth by means of experiment: we are now about to shew how the truth of the same proposition may be demonstrated, without recourse to experiment, by means of an axiom concerning force. We have pursued this course, not because it is necessary, but because it appears fitted to help the student over those difficulties which belong to the first study of the Science of Mechanics. The student has (we presume) made himself acquainted with Algebra, Geometry, and Trigonometry, before he enters upon Mechanics; but those subjects are entirely confined to the properties of *space* and *number*, and he is likely therefore to feel considerable difficulty if he is thrown at once upon the demonstration of propositions concerning that which is so new to him in its character and properties as *force*. Now it is hoped that by the study of the preceding chapters this difficulty will be obviated, and that being now thoroughly familiar with the propositions which he has to prove, he will not find any very great obstacle in the way of comprehending the proof.

2. The principle upon which we shall found the proof of the Parallelogram of Forces is this: *a force acting upon a particle may be supposed to act at any point in the line of its direction, that point being conceived to be rigidly connected with the particle.*

Thus let A be a particle, acted upon by a force P in the direction AP; take B any point in AP, then we may suppose P to act at B instead of A, provided A and B be conceived to be rigidly connected together. This is a principle the truth of which will be easily seen; it does not require any experiment to prove it, but may be regarded as an axiom, the truth of which the student cannot fail to see if he has really understood what is meant by foree. The principle may however be illustrated thus: Suppose W to be a weight hanging from a fixed point A by a fine string, the weight of which may be neglected, then there will be a certain pressure at A which will be equal to W: again let us put a tack through the string at any point B, so that the weight will hang from B instead of A, then the pressure on B will be equal to W, and therefore the same as it was at A in the former case: hence if we regard W as a force producing a pressure in the line ABW, we may say that W may be supposed to act either at A or at B.



This being premised, we shall proceed to give a demonstration of the parallelogram of forces, which has been already enunciated in p. 17; we shall find it convenient to divide the proposition into two, in the first of which we shall consider the *direction* of the resultant of two forces, and in the second its *magnitude*.

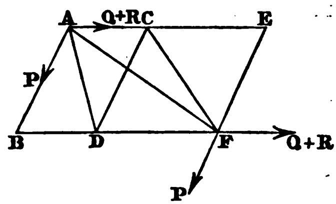
3. Prop. *If two forces, acting on a particle at A, be represented in direction and magnitude by the lines AB, AC, then the resultant will be represented in* direction *by the diagonal AD of the parallelogram described upon AB, AC.*

(1) When the forces are *equal*, it is manifest that the direction of the resultant will *bisect* the angle between the directions of the forces: or, if we represent the forces in direction and magnitude by two straight lines drawn from the point at which they act, the diagonal of the parallelogram described upon these lines will be the direction of

the resultant. Hence the proposition is true for *equal* forces.

(2) Next, suppose that the proposition, just proved for equal forces, is true for two unequal forces P and Q, and also for P and R: we shall shew that it will be true for P and $Q + R$.

Let A be the point of application of the forces; take AB to represent P in direction and magnitude, and AC to represent Q; complete the parallelogram $ABDC$, then *by hypothesis* AD is the *direction* of the resultant of P and Q; and since a force may be supposed to act at any point of its direction, we may consider D as the point of application of the resultant of P and Q; therefore, since the resultant is in all respects equivalent to its components, we may suppose the forces P and Q themselves to act at D, P parallel to AB, and Q parallel to AC; or still further we may suppose P to act at C, in the direction CD.



Again; the force R which acts at A may be supposed to act at C; take CE to represent it in direction and magnitude, and complete the parallelogram $CDFE$; then *by hypothesis*, CF is the direction of the resultant of P and R acting at C: hence the resultant of P and R may be supposed to act at F, or P and R may be supposed themselves to act at that point parallel to their original directions.

Lastly; the force Q which at present is supposed to be acting at D in the direction DF, may be supposed to act at F.

Hence we have reduced the forces P and $Q + R$, acting at A, to P and $Q + R$, acting at F; consequently F is a point in the line of action of the resultant, and therefore AF is the direction of the resultant: that is, if the proposition be true for P and Q, and also for P and R, it is true for P and $Q + R$.

But the proposition is true for P and P, and also for

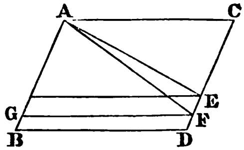
P and P, therefore it is true for P and $P+P$ or $2P$; therefore for P and $P+2P$ or $3P$; and so on; therefore generally for P and mP, where m is any whole number.

In like manner the proposition may be extended to mP and nP, m and n being any whole numbers. We may therefore consider the proposition to be true for all forces*.

∴ If two forces, &c. Q.E.D.

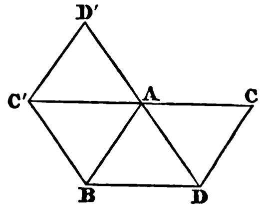
* Rather for all *commensurable* forces; that is, for all forces the ratio of whose magnitudes can be expressed by the ratio of two whole numbers. But this is not the case with all forces; for instance, we might have two forces, one measured by $\sqrt{3}$ lbs. and the other by 2 lbs.; now the ratio of $\sqrt{3}:2$, cannot be expressed by the ratio of two whole numbers *exactly*, though it can be so expressed *as nearly as ever we please*. To make this more clear, observe that $\frac{\sqrt{3}}{2} = .8660254$ very nearly; hence $\sqrt{3} : 2 :: 8660254 : 10000000$, very nearly, and by taking more decimal places we could make the approximation still more close. Now as the proposition proved in the text is true for two forces whose ratio is $8660254 : 10000000$, we should seem to be safe in concluding that it is also true for the *incommensurable* forces whose ratio is $\sqrt{3}:2$. And by considering the matter thus we might conclude that the proposition proved in the text for *commensurable* forces is true also for *incommensurable*. To take away however all kind of doubt we subjoin the following *reductio ad absurdum*.

Let AB, AC represent any two incommensurable forces; complete the parallelogram $ABDC$, and if AD be not the direction of the resultant, let it be AE. Suppose AC to be divided into a number of equal parts, each part being less than ED, and suppose distances of the same magnitude to be set off along CD, beginning at C, then one of the divisions must fall between E and D; let F be the point which marks the division, and complete the parallelogram $AGFC$, then AF is the direction of the resultant of the *commensurable* forces AG, AC: but AF makes a larger angle with AC than AE, that is, the resultant of AG and AC lies further away from AC than the resultant of AB and AC, although AG is less than AB, which is absurd: hence AE is not the direction of the resultant; and it may be shewn in like manner that no line is in that direction except AD. Hence the proposition proved in the text for *commensurable* forces, is true also for incommensurable.



4. PROP. *If two forces, acting on a particle at* A, *be represented in direction and magnitude by the straight lines* AB, AC, *then the resultant will be represented, not only in direction, but also in* magnitude, *by the diagonal* AD *of the parallelogram described upon* AB, AC.

Produce the diagonal DA to D', making AD' *equal* to the resultant of AB, AC in *magnitude;* complete the parallelogram $ABC'D'$, and join AC'.



Then since AD' is *equal* to the resultant of AB, AC, and drawn in the direction opposite to that of the resultant, the three forces AB, AC, AD', balance each other, and therefore any one of them is in the direction of the resultant of the other two; hence AC is in the direction of the resultant of AB, AD'; but AC' is also in that direction, therefore AC, AC' are in the same straight line. Hence $ADBC'$ is a parallelogram; therefore $AD = BC'$: but $BC' = AD'$, therefore $AD = AD'$. And by construction AD' represents the resultant of AB and AC in magnitude; therefore AD also represents the resultant.

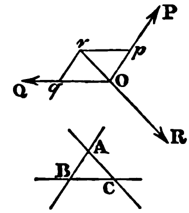
∴ If two forces, &c. Q.E.D.

5. We have thus established the proposition known as the Parallelogram of Forces, without appeal to experiment as in Chap. I.; and the proposition may now be supposed to rest upon the same kind of evidence as the theorems of Euclid.

6. The proposition is sometimes stated in a form, in which it is called the *Triangle of Forces.* We will enunciate it as follows.

PROP. *If three forces, acting in the same plane, be in equilibrium upon a particle, and if in that plane we draw any three straight lines parallel to the directions of the forces, then the three sides of the triangle so formed will be in the same proportion as the forces.*

Let O be the particle, P, Q, R the forces; upon the directions OP, OQ set off Op, Oq proportional to P and Q: complete the parallelogram $Oprq$, and join Or, then Or is in the same straight line with OR, by the parallelogram of forces; and the three lines Op (or rq,) Oq, Or are proportional to P, Q, R, respectively.



Now draw three straight lines AB, BC, AC, parallel to the directions of P, Q, R respectively, that is, parallel to rq, Oq, Or; then the triangle ABC so formed is similar to the triangle rqO;

$$\therefore AB : BC : AC :: rq : Oq : Or,$$

$$:: P : Q : R.$$

$\therefore$ If three forces, &c. Q.E.D.

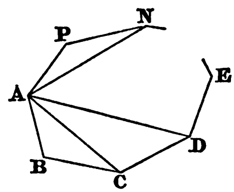
Cor. Hence, if two sides of a triangle, taken in order from an angular point, represent in magnitude and direction two forces which act at that point, then the third side, *not taken in the same order as the other two* will represent the resultant. Thus if AB, BC represent two forces acting on a particle at A, then AC (*not* CA) will represent the resultant.

7. We may generalize this proposition still further and deduce what may be called the *Polygon of Forces.*

Prop. *If the sides of a polygon* AB, BC, CD, DE, ... NP, PA, *represent in magnitude and direction forces acting upon a particle, these forces will produce equilibrium; and any one of the sides, as* AP, *taken in the opposite direction to that above supposed, will represent the resultant of all the rest.*

Join AC, then AC represents the resultant of AB, BC.

4—2



Join AD, then AD represents the resultant of AC, CD, i.e. of AB, BC, CD.

And so on: hence AN represents the resultant of AB, BC, CD, DE......

But the forces represented by AN, NP, PA, are in equilibrium; hence the forces represented by AB, BC, CD, DE,... NP, PA are in equilibrium.

Hence the first part of the proposition is true; and the second immediately follows.

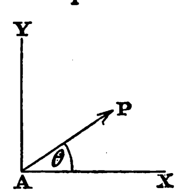
∴ If the sides of a polygon, &c. Q.E.D.

OBS. It may be remarked that the straight lines, AB, BC, &c., need not be all in the same plane.

8. The student is already acquainted with the application of the parallelogram of forces to the resolution of a force into two components in any directions (Art. 12. p. 19), and he will remember that we especially called attention to that case of resolution in which the components are at right angles to each other (Art. 14. p. 21). We shall now proceed to apply the principle of resolution and composition of forces to the following very general proposition.

PROP. *Any number of forces act at the same point, their directions all lying in the same plane; to find the direction and magnitude of the resultant.*

Let P be any one of the forces acting at the point A. Let the plane of the paper be that in which the forces act; in that plane choose any two lines at right angles to each other, AX and AY, and let θ be the angle which the direction of P makes with AX. Then P is equivalent to



$P\cos\theta$ acting in the direction AX,

together with $P\sin\theta$ . . . . AY.

In like manner, a force P', the direction of which makes an angle θ' with AX, is equivalent to

$$P' \cos \theta' \text{ acting in the direction } AX,$$

$$\text{together with } P' \sin \theta' \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad AY.$$

And so on for any number of forces. Hence, adding together the forces which act in the same direction, we shall have a system of forces $P, P', \ldots$ acting at angles $\theta, \theta', \ldots$ with the line AX, equivalent to

$$P \cos \theta + P' \cos \theta' + \text{ acting in the direction } AX,$$

$$\text{together with } P \sin \theta + P' \sin \theta' + \ldots \ldots \ldots \ldots \quad AY.$$

For shortness sake, let

$$P \cos \theta + P' \cos \theta' + \ldots \ldots = X,$$

$$\text{and } P \sin \theta + P' \sin \theta' + \ldots \ldots = Y;$$

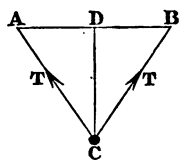
and let R be the required resultant, ϕ the angle which its direction makes with the line AX; then

$$R \cos \phi = X,$$

$$R \sin \phi = Y;$$

$$\therefore \tan \phi = \frac{Y}{X}, \quad R^2 = X^2 + Y^2.$$

9. We subjoin some examples of the process of composition described in the preceding Article.



Ex. 1. A weight of 10 lbs. is supported by two strings, each of which is 3 feet long, the ends being attached to two points in a horizontal line 3 feet apart; to find the tension of each string.

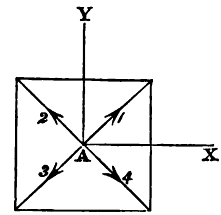
Let A, B be the two points of support; AC, BC the strings; C the weight, T the tension of either of the strings, that is the force which it exerts upon the weight in the direction of its length. Draw CD perpendicular to AB; then ABC is an equilateral triangle, and $ACD = BCD = 30^0$.

Then resolving the two tensions vertically, we have for the resolved part of each $T\cos 30^0 = T\frac{\sqrt{3}}{2}$; and the two vertical resolved parts together support the weight of 10 lbs.;

$$\therefore\ T\sqrt{3} = 10,$$

$$\text{or, } T = \frac{10}{\sqrt{3}} \text{ lbs.}$$

Ex. 2. A particle placed in the centre of a square is acted upon by forces of 1, 2, 3, and 4 lbs. respectively, tending to the angular points; to find the magnitude and direction of the resultant force.



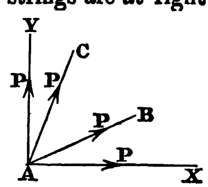
Let A be the particle, and draw two lines AX, AY perpendicular to the sides of the square, as in the figure. Then it will be seen that the application of the formulæ of the preceding article gives us the following;

$$\begin{aligned} R\cos\phi &= 1 \times \cos 45^0 + 2 \times \cos 135^0 + 3 \times \cos 225^0 + 4 \cos 315^0, \\ &= \cos 45^0 - 2\cos 45^0 - 3\cos 45^0 + 4\cos 45^0 = 0; \\ R\sin\phi &= 1 \times \sin 45^0 + 2 \times \sin 135^0 + 3 \times \sin 225^0 + 4 \sin 315^0, \\ &= \sin 45^0 + 2\sin 45^0 - 3\sin 45^0 - 4\sin 45^0 = -4\sin 45^0, \\ &= -2\sqrt{2}. \end{aligned}$$

$\therefore\ \phi = 90^0$, and $R = -2\sqrt{2}$; *i.e.* the resultant is a force of $2\sqrt{2}$ lbs. acting in the direction opposite to AY.

Ex. 3. If four equal forces act by strings upon a particle, the angles between the directions being 30^0, 45^0, and 15^0, to find the direction in which the particle will begin to move.

Let A be the particle; then the first and last strings are at right angles to each other; therefore we may conveniently take them as corresponding to the lines of reference which we have called AX and AY. Let AB, AC be the other two strings, and P the force exercised by each. Then we shall have,



$$R\cos\phi = P + P\cos 30^0 + P\cos 75^0,$$

$$R\sin\phi = P\sin 30^0 + P\sin 75^0 + P;$$

$$\therefore \tan\phi = \frac{1+\sin 30 + \sin 75^0}{1+\cos 30 + \cos 75^0} = \frac{1+\frac{1}{2}+\frac{\sqrt{3}+1}{2\sqrt{2}}}{1+\frac{\sqrt{3}}{2}+\frac{\sqrt{3}-1}{2\sqrt{2}}}$$

$$= \frac{3\sqrt{2}+\sqrt{3}+1}{2\sqrt{2}+\sqrt{6}+\sqrt{3}-1}.$$

This formula may be reduced to numbers, and ϕ determined by means of trigonometrical tables.

10. We have already enunciated in the form of the *polygon of forces* the most general conditions of the equilibrium of a system of forces acting on a particle; this may be called the *geometrical* form of the conditions of equilibrium. We shall now investigate the conditions algebraically.

Prop. *To find the conditions of equilibrium of any system of forces, acting in one plane at the same point.*

Suppose the forces to be all reduced to one (R), as in Art. 8; then in order that there may be equilibrium, we must have

$$R = 0,$$

$$\text{or } X^2 + Y^2 = 0.$$

And this equation cannot be true, unless we have

$$X = 0 \text{ and } Y = 0;$$

$$\text{or } P\cos\theta + P'\cos\theta' + \ldots = 0,$$

$$P\sin\theta + P'\sin\theta' + \ldots = 0.$$

These are the conditions of equilibrium; and they may be expressed in words by saying, *that the sum of the forces resolved in any two directions perpendicular to each other must vanish.*

11. We shall now illustrate these principles of equilibrium by applying them to several examples.

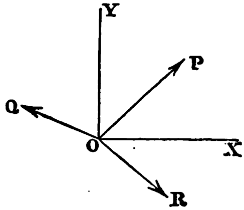
Ex. 1. If three forces P, Q, R, be in equilibrium upon a point O; then

$$P : Q : R :: \sin QOR : \sin ROP : \sin POQ.$$

This immediately follows from the *triangle of forces*. For referring to the figure of Art. 6, we have

$$\begin{aligned} P : Q : R &:: AB : BC : AC, \\ &:: \sin ACB : \sin BAC : \sin ABC, \text{ by Trigonometry} \\ &:: \sin QOR : \sin ROP : \sin POQ. \end{aligned}$$

It will, however, be worth while to deduce the result from the principle of the preceding Article.



Draw any two straight lines OX, OY at right angles to each other, and let $XOP = \theta$, $XOQ = \phi$, $XOR = \psi$, these angles being all measured the same way round from OX.

Then, in order that P, Q, R may be in equilibrium, we must have

$$P \cos\theta + Q\cos\phi + R\cos\psi = 0,$$
$$P\sin\theta + Q\sin\phi + R\sin\psi = 0.$$

Multiply these equations by $\sin\psi$ and $\cos\psi$ respectively, and subtract; then we have,

$$P(\sin\psi\cos\theta - \sin\theta\cos\psi) + Q(\sin\psi\cos\phi - \sin\phi\cos\psi) = 0,$$

or, $\quad P\sin(\psi - \theta) + Q\sin(\psi - \phi) = 0$;

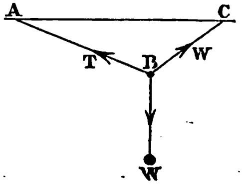
but $\quad \psi - \theta = 360^\circ - ROP$, and $\psi - \phi = QOR$,

$$\therefore\ -P\sin ROP + Q\sin QOR = 0,$$

$$\text{or, } \frac{P}{\sin QOR} = \frac{Q}{\sin ROP}.$$

Hence we may conclude that

$$\frac{P}{\sin QOR} = \frac{Q}{\sin ROP} = \frac{R}{\sin POQ}, \text{ as before.}$$



Ex. 2. A small ring B is attached to the extremity of a thread AB, which is fastened at A. CBW is another thread passing through the ring B and supporting a weight W. To find the position of B; A and C being in *the same horizontal line*.

We are to regard the ring B as a particle, kept at rest by three forces acting in the directions of the three portions of thread which meet in it. Concerning the force exerted by AB, in other words the *tension* of the thread, we know nothing, we will therefore denote it by a symbol T; the length of AB is given, call it l. With regard to the other thread, we observe that the force exerted by the upper portion of it, CB, must be equal to that exerted by the lower portion BW, in other words, the *tension* is the same throughout the same thread; therefore we shall have a force W in the direction BC, and another force W in the direction BW. The distance AC must be given, call it a. And let $BAC = \theta$, $ACB = \phi$.

Then, resolving the forces horizontally and vertically, we have the two following equations;

$$T \cos\theta - W\cos\phi = 0, \qquad \text{..................(1)}$$

$$T \sin\theta + W\sin\phi - W = 0, \qquad \text{..................(2)}$$

But these equations involve *three* unknown quantities, T, θ, and ϕ, therefore we must have *one other* relation between them; this is supplied by the trigonometrical conditions of the triangle ABC; for we have

$$\frac{l}{a} = \frac{\sin\phi}{\sin(\theta+\phi)} \qquad \text{........................(3)}$$

Now multiplying (1) by $\sin\theta$, and (2) by $\cos\theta$, and subtracting, there results

$$W(\sin\theta\cos\phi + \cos\theta\sin\phi) - W\cos\theta = 0,$$

$$\text{or,} \quad \sin(\theta+\phi) = \cos\theta,$$

$$\therefore \ \theta + \phi = 90^0 - \theta,$$

$$\text{or,} \quad \phi = 90^0 - 2\theta \qquad \text{.....................(4)}$$

$$\therefore \text{ from (3)}, \ \frac{l}{a} = \frac{\cos 2\theta}{\sin\theta},$$

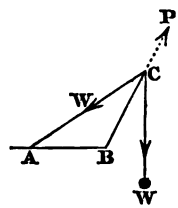
$$1 - 2\sin^2\theta = \frac{l}{a}\sin\theta,$$

$$\sin^2\theta + \frac{l}{2a}\sin\theta + \frac{l^2}{16a^2} = \frac{l^2}{16a^2} + \tfrac{1}{2},$$

$$\sin\theta = -\frac{l}{4a} \pm \sqrt{\frac{l^2}{16a^2} + \tfrac{1}{2}}.$$

The + sign must be taken for the radical, since it is evident that θ must be less than 180^0. The value of θ thus found entirely determines the *position of B.*

Ex. 3. A rod BC is moveable in a vertical plane about a hinge at B; a thread, attached to a point A in the same horizontal line as B, passes over the extremity of the rod and supports a weight W. Omitting the weight of the rod, it is required to find the position of equilibrium.



In this problem we must regard the extremity C of the rod as a particle, which is kept in equilibrium by the two forces exerted by the thread in the directions CA and CW respectively, and by that which the rod itself exerts in the direction of its length. The first two forces will be each equal to W; the last we will denote by P.

Then if $AB = a$, $BC = b$, $BAC = \theta$, $ACB = \phi$, we have, by resolving the forces horizontally and vertically, the following equations,

$$W \cos\theta - P\cos(\theta + \phi) = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$W \sin\theta - P\sin(\theta + \phi) + W = 0; \quad \ldots\ldots\ldots\ldots(2)$$

and we have also the Trigonometrical condition,

$$\frac{a}{b} = \frac{\sin\phi}{\sin\theta}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Having obtained these three equations, θ, ϕ, and P may all be found, and the problem may be completed in the same manner as the preceding one.

CONVERSATION UPON CHAPTER IV.

P. Is the proof which you have given in this chapter of the Parallelogram of Forces the only one which has been devised?

T. Far from it, there are a great variety*; and many

* "We have to ask whether this proposition, the Parallelogram of Forces, be a necessary truth; and if so, on what grounds its necessity ultimately rests. We shall find that this, like the other fundamental doctrines of Statics, justly claims a demonstrative certainty. Daniel Bernoulli, in 1726, gave the first proof of this important proposition on pure Statical principles; and thus, as he says, 'proved that statical

of them are of a kind which your present knowledge of Mathematics will not enable you to comprehend. That which I have given you seems upon the whole to be the most easy of comprehension; it is called from its inventor Duchayla's proof. The frequent transfer of the forces from one point to another which it involves is rather perplexing at first, but after having written it upon paper several times, and represented by a figure the transfer of the forces, you will find little difficulty in understanding and remembering it.

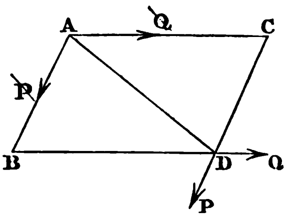
P. What do you mean by representing by a figure the transfer of the forces?

T. Suppose P and Q to be two forces acting at A;

theorems are not less necessarily true than geometrical are.' If we examine this proof of Bernoulli, in order to discover what are the principles on which it rests, we shall find that the reasoning employs in its progress such axioms as this:—That if from forces which are in equilibrium at a point be taken away other forces which are in equilibrium at the same point, the remainder will be in equilibrium; and generally,—That if forces can be resolved into other equivalent forces, these may be separated, grouped, and recombined, in any new manner, and the result will still be identical with what it was at first.... The apprehension of force as having magnitude, as made up of parts, as capable of composition, leads to axioms in Statics, in the same manner as the like apprehension of space leads to the axioms of Geometry. And thus the truths of Statics, resting upon such foundations, are independent of experience in the same manner in which geometrical truths are so.

The proof of the parallelogram of forces thus given by Daniel Bernoulli, as it was the first, is also one of the most simple proofs of that proposition which have been devised up to the present day. Many other demonstrations, however, have been given of the same proposition. Jacobi, a German mathematician, has collected and examined eighteen of these. They all depend either upon such principles as have just been stated: That forces may in every way be replaced by those which are equivalent to them; or else upon those previously stated, the doctrine of the lever, and the transfer of a force from one point to another of its direction. In either case, they are necessary results of our statical conceptions, independent of any observed laws of motion, and indeed of the conception of actual motion altogether."—Whewell's *Philosophy of the Inductive Sciences*, Bk. III. chap. vi.

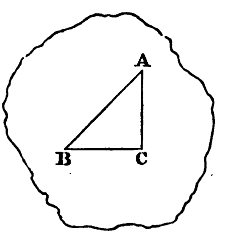
then if we complete the parallelogram $ABDC$, we have seen that P and Q may be supposed to act at D parallel to their original directions. Now to denote this I should draw arrows from D to represent P and Q, and then draw my pen through the symbols of the forces at A; and my figure will thus represent to the eye the exact state of the transfer of the forces. When I transfer P from D to C, I should draw an arrow at C, and put my pen through the P at D; and so on.



P. That seems a useful hint. I suppose I need not trouble myself with any of those other proofs of the parallelogram of forces to which you have alluded.

T. That of Duchayla is sufficient for all purposes; nevertheless it will tend to assist your understanding of the subject, if I point out to you occasionally other methods by means of which the principles of Statics may be established. I will, for instance, shew you now a method of establishing the rule for the resolution of forces, which is highly ingenious; it is similar to the method adopted by Stevinus, the famous mathematician of Bruges, which I shall at a future time bring under your notice.

Let ABC be a groove, cut in the vertical side of a block of any hard substance; let its form be that of a right-angled triangle, AC being vertical, and BC horizontal. The groove must be conceived to be of perfectly uniform section throughout, and perfectly smooth.



Now suppose that we have an endless filament or thread which exactly fits the groove ABC. Then if it were fitted into ABC it would remain at rest exactly in the position in which we placed it; and this entirely independent of all friction or any such impediments to motion. For suppose it moved

into a new position; then in this new position all the circumstances of the thread would be precisely the same as they were when we first placed it in the groove; therefore as it moved before it will move now, and so it will continue to move, or the motion will be perpetual, which is evidently impossible. Hence the thread will remain at rest; the same would be the case whatever were the shape of the groove, but I have taken the form of a right-angled triangle for reasons which will appear immediately.

Let us consider the equilibrium of the thread: the portion BC, resting as it does upon the horizontal surface of the groove, can have no effect in pulling the string downwards either at B or C. Hence we may if we please suppose the portion BC of the thread removed, and we have then the portion AC which is altogether unsupported by the groove (for it hangs vertically) in equilibrium with the longer portion AB which is partly supported by resting upon the side AB of the groove.

Since the thread is uniform the weights of the portions AB, AC are proportional to the lines AB, AC respectively. Also the weight of AC measures the *resolved part* of the weight of AB in the direction of the groove AB, because it is that which supports AB;

$$\therefore \text{ resolved part of weight of } AB$$
$$: \text{weight of } AB :: AC : AB,$$
$$\text{or, resolved part of weight of } AB$$
$$= \text{weight of } AB \times \frac{AC}{AB},$$
$$= \text{weight of } AB \times \cos BAC.$$

The result would be precisely the same, if instead of having the thread AB resting upon the groove we had a weight W at any point; and we should therefore have

$$\text{resolved part of } W \text{ along } AB = W \cos BAC.$$

This is the law of resolution which we have already obtained; and we might therefore, if we pleased, take this demonstration as the basis of the science of Statics.

P. Could the process which you have described be verified experimentally?

T. Assuredly not: for none of the conditions could be accurately satisfied: the groove could not be made *quite* smooth, and *quite* uniform, nor the thread *perfectly* flexible and *perfectly* to fit the groove. But all these conditions may be *conceived* to be satisfied, and the conclusion rests upon the axiom of the impossibility of the thread, when placed in the groove at rest, beginning to move of itself and continuing to revolve perpetually. If you grant this axiom, then the conclusions are perfectly sure, although the construction described may be (as it is) quite beyond human art.

P. Would not the principle which you explained to me when speaking of the centre of gravity, I mean the principle of the centre of gravity always tending to fall as low as possible serve to shew that the thread must rest in a certain position?

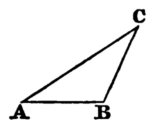
T. You may regard it in that light; the tendency of the weight of the string, that is, of the earth's attraction, is to depress the centre of gravity, and as the centre of gravity cannot be depressed on account of the string being held in a certain shape, therefore the string will not move.

P. Could not the parallelogram of forces be demonstrated by reference to the manner in which a particle would *move* if acted upon by two forces?

T. There are proofs depending upon this principle, but they do not properly belong to the subject which we are at present studying: we are engaged, you will remember, with *Statics,* or the science of forces which produce *equilibrium,* and therefore any considerations of force producing *motion* belongs to another branch of the science. At the same time one or two remarks arising from the question you have asked may be instructive. You will

see in the chapter which you have just been reading, that if it can be shewn that the parallelogram of forces is true so far as *direction* is concerned, it can be immediately proved to be true as regards magnitude (page 74). Hence we need not concern ourselves with anything except the *direction* of the resultant of two forces, or the direction in which a particle acted upon by those two forces *would begin to move.*

Suppose now that we have a particle at A, and suppose that a certain force acting upon it would in a given short space of time, as one second, carry it to B; suppose also that when it has been carried to B another force acts upon it, which carries it in one second to C. Now if



we may regard the action of two forces upon a particle during one second as equivalent to the action of one force for one second by itself, and the action of the other force for a second by itself, then the two forces combined would in one second carry the particle from A to C, that is, AC would be the direction of the particle's motion or the direction of the resultant of the two forces. In this manner we should be able to prove the parallelogram of forces so far as regards direction, and then, as I have already mentioned, the proposition may be easily proved as regards magnitude. But this method involves a law of force to assume which would be to anticipate the laws of motion, and the method of proof involving as it does considerations of motion is not suitable for a statical treatise. Newton (it is true) in the commencement of the *Principia* treats of the resolution and composition of forces in this way, but then it is in connexion with laws of motion which he has previously enunciated, and as introductory to a work which is concerned with Dynamics. On the whole I should recommend you to adopt Duchayla's proof as the most simple; but you will find that the study of any other which may fall in your way will help to clear your view of the subject in general, and I shall take occasion myself to call your attention to a method which in many respects

is extremely simple and which depends upon the Principle of the Lever, which is according to that system made the fundamental proposition of the science. This however I can do with more propriety when you have studied the next chapter.

P. I now wish to ask for some explanation concerning the problem given in p. 81. I do not clearly understand how equations (1) and (2) are obtained.

T. Conceive a horizontal line drawn through B to the right hand; then, measuring the angles the same way round from this line, the angle which it makes with BC will be ϕ, with BA $180^0 - \theta$, with BW 270^0; hence the two equations of Art. 10 will become

$$W \cos \phi + T \cos (180^0 - \theta) + W \cos 270^0 = 0,$$
$$W \sin \phi + T \sin (180^0 - \theta) + W \sin 270^0 = 0;$$

which are equivalent to

$$W \cos \phi - T \cos \theta = 0,$$
$$W \sin \phi + T \sin \theta - W = 0;$$

as given in the place to which you refer. And in all cases the mechanical equations of a problem may be obtained in the same way; you may, however, in general obtain them more simply by resolving each of the forces into two, taking for the angle of resolution the acute angle which the direction of the force makes with the assigned direction, and then consider whether any two resolved parts, horizontal parts for instance, tend to move the particle in the same or in opposite directions; those which tend in one direction must be first written down, and then those which tend in the opposite direction must be affected with the negative sign.

For example, in our present problem, the thread AB produces a horizontal force $T \cos \theta$, and a vertical force $T \sin \theta$. The thread BC produces a horizontal force $W \cos \phi$, and a vertical force $W \sin \phi$. Now it is easy to see that these horizontal portions tend to draw the ring in

opposite directions, and they are the only horizontal forces, therefore we must have

$$T \cos \theta = W \cos \phi,$$

$$\text{or} \quad T \cos \theta - W \cos \phi = 0.$$

And the two vertical portions both act upwards, and together support the weight W; therefore

$$T \sin \theta + W \sin \phi = W,$$

$$\text{or} \quad T \sin \theta + W \sin \phi - W = 0,$$

and thus we obtain the equations required.

P. I think that by help of what you have said I shall be able to overcome my difficulty: why did you call the equations of which we have been speaking *mechanical* equations?

T. The equations with which you will have to deal in this subject will be of two kinds; those which arise from the two mechanical principles, namely, the Parallelogram of Forces, and the Principle of the Lever, and those which arise from the necessary geometrical connexion of the different parts of the system. In considering the equilibrium of a particle, acted upon by forces whose directions lie all in one plane, the equations which result from the Parallelogram of Forces are *two* and *only two;* and these we call the *mechanical* equations of the problem. If these equations involve only *two* unknown quantities, they contain the complete solution of the problem; but if, as is frequently the case, they contain more than two, then other relations between the unknown quantities must be sought from geometrical considerations; the equations so found, which of course contain no forces, but only lines and angles, are called *geometrical equations.* And let me here remark, that in the solution of problems it is always necessary to obtain as many equations as there are unknown quantities involved; so that if there be n unknown quantities, you must, before you can solve the problem, obtain $n - 2$ geometrical relations among them.

P. I observe that Ex. 1, on p. 80, is solved in two different ways, one of which seems much shorter than the other; is there more than one way of solving Ex. 2?

T. The greater number of statical problems may be solved in more than one way. The advantage of the general method given in p. 79 is, that it includes all kinds of problems, that it is simple in its principle and easily applicable in almost all cases. At the same time it must be allowed, that many problems may be solved more concisely by choosing methods peculiarly suitable to them. In the present instance, the equation (4) of p. 81, which together with the geometrical equation (3) contains the solution of the problem may be obtained readily thus:

The tension T is in the direction of the resultant of the two tensions which act in the directions BC, BW. But these two latter tensions are equal; therefore AB must bisect the angle between BC and BW.

Now $WBC = 90^\circ + \phi$;

and by our principle, $\dfrac{WBC}{2} = BAC + BCA$ (Euc. I. 32).

$$\therefore\ 90^\circ + \phi = 2\theta + 2\phi,$$

$$\text{or } \phi = 90^\circ - 2\theta,$$

which is the equation required.

The same method will simplify the solution of Ex. 3, p. 82. And you will probably meet with other cases in which a little ingenuity will save you much trouble; but I recommend you on no account to neglect the application of the uniform general method of p. 79, which, though not always the shortest, is certainly the surest. Sometimes a geometrical construction will be able to take the place of elimination between several equations, and when you become familiar with the subject, you may adopt in each problem the method which seems to you best; but you must remember that we are studying *Mechanics* and not *Geometry*, and therefore those methods are the most

important and the best, which exhibit from the clearest point of view the mechanical conditions of the problem. There will be more scope for problems, and you will be able to see the bearing of different methods better, when we have studied the conditions of equilibrium of a rigid body.

EXAMINATION UPON CHAPTER IV.

1. PROVE the parallelogram of forces, so far as the *direction* of the resultant is concerned, for *commensurable* forces.

2. Extend the proof to the case of *incommensurable* forces.

3. Assuming the parallelogram of forces so far as the *direction* of the resultant is concerned, prove it as to *magnitude*.

4. Enunciate the *Triangle of Forces*.

5. Enunciate the *Polygon of Forces*.

6. Determine algebraically the direction and magnitude of the resultant of any number of forces acting in given directions at the same point, the directions being supposed to lie all in one plane.

7. Investigate algebraically the conditions of equilibrium of a particle under the action of any forces whose directions all lie in one plane.

8. The resultant of two forces which act at right angles to each other is equal to n times the geometrical mean between them; find the ratio of the two forces, and the smallest value of n for which the problem is possible.

9. Given the sum of two forces, and their resultant when they act at an angle of $60°$ with each other; find the forces.

10. A and B can each carry a weight of P lbs. What weight can they carry between them, when walking a feet apart, by means of two cords, each b feet long, attached to the weight?

11. Two equal weights (W) are attached to the extremities of a thread, which is suspended from three tacks in a wall, forming an equilateral triangle; find the pressure on each tack.

12. In the preceding problem find the *vertical* strain upon each tack, supposing the base of the triangle to make an angle θ with the horizon.

13. A particle at the centre of a regular hexagon is urged towards the six angular points by forces equivalent to 1, 2, 3, 4, 5, 6 lbs. respectively; determine the direction and magnitude of the resultant.

14. A, B, and C pull at the ends of three ropes which are knotted together in O. B and C are of equal strength, and A is as strong as B and C together; what help will B and C require to maintain O in equilibrium against A when $BOC = 60°$?

15. A fine thread has a small ring at one extremity; the other extremity is passed through the ring and attached to a weight; the whole is suspended by means of the loop thus formed from two smooth tacks in the same horizontal line; determine the position of equilibrium.

16. In the preceding problem find the direction and magnitude of the pressure upon the tacks.

17. A given force R is divided into two others P and Q $(P + Q = R)$; prove that the resultant of P and Q, supposed to act on a point at right angles to each other, will be least when $P = Q$.

18. Six men pull by means of a rope 100 feet long attached to the top of a tree 60 feet high towards the South; and five men by means of a rope 12 feet long towards the East; find in what direction the tree will fall.

CHAPTER V.

DEMONSTRATIVE MECHANICS. PRINCIPLE OF THE LEVER. THEORY OF COUPLES. CONDITIONS OF EQUILIBRIUM OF A RIGID BODY, THE DIRECTIONS OF THE FORCES BEING ALL IN ONE PLANE.

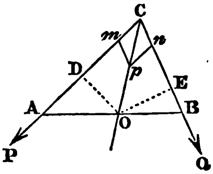
---

1. In the preceding Chapter we have been concerned entirely with the equilibrium of forces acting on *a particle*, or the conditions under which *a particle* acted upon by any system of forces whose directions are in one plane *can be at rest*. In the present we shall be occupied with the conditions of equilibrium of a *rigid body;* we have already, in Chap. II., considered the particular case of two weights suspended upon a lever, and we shewed, experimentally, that the condition of equilibrium was the equality of the moments of the two weights about the fulcrum: we shall now shew how this and some more general results may be deduced from the Parallelogram of Forces, which in the preceding Chapter we have demonstrated.

We shall commence with the *Principle of the Lever.*

2. Prop. *If two forces acting at the extremities of a lever, and tending to twist it opposite ways, produce equilibrium, the moments of the forces about the fulcrum are equal.*

I. Let the directions of the forces be not parallel.



Let P and Q be the forces, acting at the extremities A, B, of the lever AB. Produce the directions of P and Q until they meet in C; then P and Q may both be supposed to act at C. Take Cm, Cn, proportional to P and Q,

and complete the parallelogram $Cmpn$; join Cp and produce it to cut AB in O, then the resultant of P and Q acts in the direction CO, and therefore O must be the fulcrum, otherwise there could not be equilibrium. Draw OD, OE perpendicular to AC, BC; then

$$\frac{P}{Q} = \frac{Cm}{Cn} = \frac{Cm}{mp} = \frac{\sin Cpm}{\sin mCp} = \frac{\sin nCp}{\sin mCp}$$

$$= \frac{OC \sin OCE}{OC \sin OCD} = \frac{OE}{OD};$$

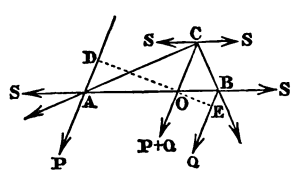
$$\therefore \; P.OD = Q.OE,$$

or the moments of P and Q about O are equal.

II. Let the directions of the forces be parallel.

In this case the directions of the forces when produced, will not meet in a point, as supposed in the preceding investigation; and we must therefore slightly modify the method.

At A and B apply any two equal and opposite forces S in the direction of the lever; this will manifestly not affect the equilibrium; then the resultant of P and S will be a force in the direction CA, suppose, and that of Q and S a force in the direction



CB. Suppose them both to act at C, and there to be resolved into their constituent parts P and S, Q and S; the portions S, S will destroy each other, leaving a resultant $P+Q$ in the direction CO parallel to the directions of the forces.

Then the sides of the triangle AOC are parallel to the directions of the forces P, S and their resultant; therefore by the *triangle of forces* (Art. 6. p. 74),

$$\frac{P}{S} = \frac{CO}{AO}.$$

In like manner $\qquad \frac{Q}{S} = \frac{CO}{BO};$

$$\therefore P.AO = Q.BO.$$

If the forces P and Q be perpendicular to the lever, this formula proves the proposition; if not, from O draw OD, OE perpendicular to the directions of P and Q; then the triangles AOD, BOE being similar, we have

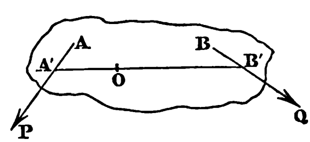
$$\frac{AO}{OD} = \frac{BO}{OE};$$

$$\text{and } \therefore P.OD = Q.OE.$$

Hence *If two forces,* &c. Q.E.D.

3. We have in the preceding demonstration supposed that the two forces act at the extremities of a *straight rigid rod;* but it is not difficult to see, that the proposition is true of two forces acting in the same plane upon any rigid body one point of which is fixed.

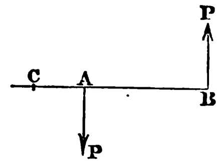
For let P and Q be two forces acting at the points A and B of a rigid body, in which the point O is fixed. Through O draw any straight line $A'OB'$, meeting the directions of P and Q in A' and B' respectively; then P may be supposed to act at A', and Q at B', and thus the problem is reduced to that of two forces acting at the extremities of the straight lever $A'OB'$.



4. We shall now proceed to the general problem of the equilibrium of any number of forces, acting in the same plane upon a rigid body. The most elegant method of treating the problem, and the simplest, is that which depends upon the properties of *couples,* which we must therefore in the first place explain.

5. Def. Two equal and opposite forces acting at right angles to a rigid rod are called *a couple.*

Let the two forces P, P act in opposite directions upon the extremities of AB, and perpendicularly to its length, then AB is called the *arm* of the couple, and $P.AB$ is called its *moment*.



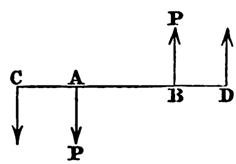
6. Now the peculiarity of a couple is this, that *it is the only case of two forces acting upon a lever, in which it is impossible to find a third force which will with the other two produce equilibrium.* If possible let C be a point in the direction of AB produced, at which a force may be applied which shall be in equilibrium with the two forces of the couple. Then by what has been already proved (Art. 2.) we must have

$$P.AC = P.BC,$$

$$\text{or } AC = BC;$$

which is impossible.

The same truth may be seen from the following simple consideration. Suppose a force applied at C to be capable of keeping the system in equilibrium; then producing AB and making $BD = AC$, a force applied at D, in the direction opposite to that which we supposed applied at C, will be situated exactly in the same manner with reference to the couple as that at C: so that if a force at C can keep the system in equilibrium, an opposite force applied at D can do the same, which is absurd.

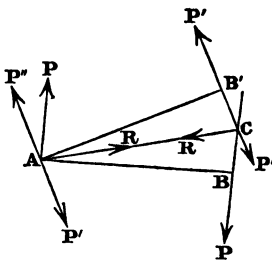


Hence we may conclude, that the effect of a couple upon a rigid rod will be to tend to make the rod *twist* about its middle point.

7. The application of the method of couples to the investigation of the conditions of equilibrium of a rigid body depends upon the three following propositions.

8. Prop. *The effect of a couple is not altered by turning its arm about one extremity through any angle in the plane of the forces.*

Let P, P be the forces, AB the arm of the couple: through A draw AB' equal to AB, and making any angle with it: at A apply two opposite forces, in the direction perpendicular to AB', and each equal to P; we shall call them P' and P'' for distinction's sake, but it will be borne in mind that they are each of the same magnitude as P. At B', in like manner, apply the equal and opposite forces P', P'', as represented in the figure. Produce the directions of P at B, and P' at B', to meet in C; then P, P' may be supposed to act at C; join AC.



Now in the triangles BAC, $B'AC$, we have $AB = AB'$, and AC common, and the right angle ABC = the right angle $AB'C$; $\therefore$ $BC = B'C$, and the triangles are equal in all respects.

Hence AC bisects the angle between the two equal forces P, P'; and therefore P, P' acting at C will have a resultant, (R suppose), in the direction CA.

Again, since PA is parallel to PC and AC meets them, the angle $PAC = PCA$: in like manner the angle $P'AC = P'CA$; hence the forces P, P' acting at A will have a resultant R, in the direction AC.

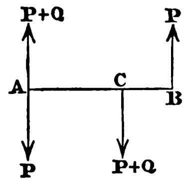
The two forces R, R, acting at A and C, in opposite directions, will neutralize each other; and thus the only forces left are P'' at A, and P'' at B'. That is, the couple with the arm AB has been transformed into a couple with the equal arm AB', and equal forces.

$\therefore$ *The effect of a couple, &c.* Q.E.D.

9. Prop. *The effect of two couples, the arms of which have a common extremity, and which tend to twist in the same direction, is the same provided their* moments *are equal.*

Let AB be the arm of a couple; P, P the forces.

At A apply a force greater than P, $P+Q$ suppose; and at C, a point between A and B, apply the force $P+Q$ in the opposite direction.



Then the two opposite forces $P+Q$ and P, acting at A, will be equivalent to a force Q acting in the direction of the former; and by what has been proved in Art. 2, the force Q at A, and the force P at B will be in equilibrium with the opposite force $P+Q$ at C, provided

$$Q.AC = P.BC,$$

$$\text{or } (P+Q)AC = P(AC+BC) = P.AB.$$

Hence the original couple will be entirely counteracted by the new couple which we have applied, which has the same moment and tends to twist in the opposite direction.

$\therefore$ *The effect of two couples, &c.* Q.E.D.

10. Taking this proposition in conjunction with the last, we see that *if one extremity of the arm be given the effect of a couple depends entirely upon its moment;* hence it is not unusual to denote a couple by its moment; thus if we have a couple of which the forces are P, P, and the arm a, we should call it *the couple* $P.a$.

11. A couple which is equivalent to any number of couples is called the *resultant* of those couples; and those couples are called with reference to that resultant *component couples.*

12. The algebraical sign $-$, which we have found useful, as designating the direction of a force, may also be applied with advantage to couples: thus, if we have two couples, one of which tends to twist a body in one direction and the other in the opposite, we may distinguish them by the signs $+$ and $-$ attached to their moments. In the preceding proposition, for example, we obtained the result,

$$(P+Q)AC = P.AB,$$

$$\text{or } (P+Q)AC - P.AB = 0.$$

If we agree to call one of these moments positive and the other negative, we shall have this result,

the algebraical sum of the moments $= 0$.

This result we shall generalize and further elucidate by the following proposition.

13. PROP. *The resultant of any number of couples, the arms of which have a common extremity, is that couple which has for its moment the algebraical sum of the moments of the component couples.*

Let Pa, $P'a'$, $P''a''$,......be the couples; and let us reduce all the couples to the arm a; thus the couple $P'a'$ will be equivalent to a couple having an arm a, and force $P'.\dfrac{a'}{a}$, since $P'a' = P'\dfrac{a'}{a}.a$, (Art. 9); and $P''a''$ will be equivalent to a couple having an arm a, and force $P''\dfrac{a''}{a}$; and so on.

Now suppose R to be the resultant of the forces acting at either end of the arm a, when the couples have been all reduced to that arm;

$$\therefore\ R = P + P'.\frac{a'}{a} + P''.\frac{a''}{a} + \ldots\ldots$$

And by the process adopted, the couples are all reduced to one, having an arm a and force R;

$$\therefore \text{ the moment of the resultant couple} = R.a,$$

$$= Pa + P'a' + P''a'' + \ldots\ldots$$

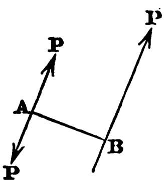
If the couples should not all tend to twist in the same direction, the moments of those which tend in the direction opposite to Pa will be negative.

$\therefore$ *The resultant, &c.* Q.E.D.

14. The three propositions which have been proved in Arts. 8, 9, 13, contain (as was announced) all the neces-

sary properties of couples, when we consider the action of forces in one plane only. We shall now apply the theory of couples to the investigation of the conditions of equilibrium of a rigid body, the direction of the forces lying all in one plane.

15. Prop. *Any system of forces, (the directions of which lie in one plane,) acting upon a rigid body, may be reduced to a single force, and a single couple.*



Let P be any one of the forces, acting in the direction BP. Take any point A in the plane of the forces; and at A apply two equal and opposite forces P, parallel to BP; this will not affect the condition of the body. Draw AB perpendicular to BP. Then instead of the force P acting in direction BP we have now the force P acting at A parallel to BP, and the couple $P.AB$.

In like manner, each of the forces may be reduced to a force at A parallel to its direction, and a couple the arm of which has A for one extremity.

Now all the *forces* at A are equivalent to *one resultant force* (Art. 8, p. 76); and all the *couples* the arms of which terminate in A are equivalent to *one resultant couple* (Art. 13).

∴ *Any system, &c.* Q.E.D.

16. It is easy to see, that the resultant force spoken of in the preceding proposition will be the same wherever the point A is taken; for if P be any one of the forces, θ the angle which its direction makes with any given straight line through A, P may be resolved into $P\cos\theta$ parallel to that line, and $P\sin\theta$ perpendicular to it (Art. 8, p. 76); and other forces P', P''... may be resolved in like manner: hence if R be the resultant force, and ϕ the angle which

its direction makes with the line from which θ is measured, we have

$$R \cos \phi = P \cos \theta + P' \cos \theta' + \ldots$$

$$R \sin \phi = P \sin \theta + P' \sin \theta' + \ldots$$

which results are altogether independent of the position of A. But in calculating the moment of the resultant couple we must find the algebraical sum of the *moments of the forces with respect to A*: thus if $AB = a$, and the perpendicular distance from A upon the direction of P' be a', and so on, we have

$$\text{moment of resultant couple} = Pa + P'a' + \ldots\ldots;$$

and this quantity manifestly depends for its value upon those of a, a' ... that is, upon the position of A.

17. From the preceding proposition we can at once deduce the *Conditions of equilibrium for a rigid body*. For we have already shewn that a force and a couple cannot counteract each other (Art. 6); hence, if a system of forces be reduced to one resultant force, and one resultant couple, the two must *separately vanish*, that is, we must have

$$\text{resultant force} = 0,$$

$$\text{moment of resultant couple} = 0.$$

The former of these conditions divides itself into two; for, (as in Art. 10, p. 79,) if a force $= 0$, each of its components must $= 0$. Hence according to the notation adopted in the preceding Article, we shall have for the conditions of equilibrium of a rigid body

$$P \cos \theta + P' \cos \theta' + \ldots\ldots = 0, \ldots\ldots (1)$$

$$P \sin \theta + P' \sin \theta' + \ldots\ldots = 0, \ldots\ldots (2)$$

$$Pa + P'a' + \ldots\ldots\ldots\ldots\ldots = 0, \ldots\ldots (3).$$

The equations (1) and (2) may be called the equations of equilibrium as regards *translation*, and are identical with those which hold for a single particle; equation (3)

may be called the equation of equilibrium as regards *twisting* or *rotation*, and is peculiar to the case of a rigid body.

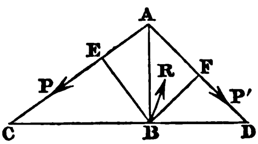
18. It is worthy of remark, that if a rigid body be capable of motion only about a certain fixed axis, the *three equations* of the preceding article are reduced to *one*; for in this case any tendency to translation will be counteracted by a pressure on the axis, and the sole condition of equilibrium will be that the resultant moment of the forces about the axis shall be zero. Nevertheless we may in this case apply the three equations, if we desire, not only to determine the position of equilibrium for the body, but also to determine the pressure upon the axis: for let R be the pressure upon the axis, ϕ the angle which the direction of R makes with the line from which θ, θ'...are measured; then the equations (1), (2) of the preceding article will become

$$R\cos\phi + P\cos\theta + P'\cos\theta' + \ldots = 0,$$

$$R\sin\phi + P\sin\theta + P'\sin\theta' + \ldots = 0,$$

and these will determine both R and ϕ.

19. We shall defer the full application of the equations of equilibrium until we have discussed, as we propose to do in the next chapter, what are called the Mechanical Powers, or the simplest cases of Machines; these might be considered merely as examples of the principles of this and the preceding chapter, but it will be convenient to group together (as is usual) in one chapter those problems which have a practical bearing, and then to collect in another such examples as may be considered chiefly theoretical and only useful as illustrations of Mechanical Principles. We shall however illustrate the meaning of the equations of this chapter by a few simple applications.



Ex. 1. AB is a vertical post moveable about a hinge at B; two men at C and D pull at the post by means of cords attached at A; given the height of the post and the lengths of the cords, compare the strengths of the men when AB remains vertical.

Let $AB=p$; $CA=l$; $DA=l'$; P, P' the forces exerted by the two men. Draw BE, BF perpendicular to AC, AD respectively; then for equilibrium we must have

$$\text{moment of } P \text{ about } B = \text{moment of } P' \text{ about } B,$$

$$\text{or } P.BE = P'.BF;$$

but by similar triangles ABC, BEC,

$$\frac{AB}{AC} = \frac{BE}{BC};$$

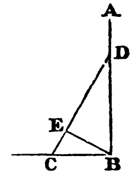
$$\therefore\ BE = \frac{AB.BC}{AC} = \frac{p}{l}\sqrt{l^2-p^2},$$

$$\text{similarly, } BF = \frac{p}{l'}\sqrt{l'^2-p^2},$$

$$\therefore\ \frac{P}{P'} = \frac{l}{l'}\sqrt{\frac{l'^2-p^2}{l^2-p^2}}.$$

This formula gives us the ratio required.

Ex. 2. We see from the preceding investigation that the effect which a man can produce, by means of a rope attached in the manner described, is measured by the *moment* of the force which he exerts, not by the force itself. Let us illustrate this by inquiring under what circumstances a man can with the greatest advantage pull at a tree AB, by means of a rope of given length CD, attached to a point D in the tree.



Let F be the whole force which the man can exert; $CD=l$; $DCB=\theta$; draw BE perpendicular to CD; then the *moment of* F *about* B

$$= F \times BE = F \times BC \sin\theta,$$

$$= F \times l\cos\theta\sin\theta = \frac{Fl}{2}\sin 2\theta.$$

Now $\sin 2\theta$ has its greatest value when $2\theta = 90°$, or $\theta = 45°$. In this case $BC = BD$; or the man will pull to the greatest advantage, when the height of the point of attachment of the rope is equal to the man's distance from the tree: and the moment produced will be equal to that which would support the greatest weight the man can lift, suspended from the extremity of a rigid rod half as long as the rope.

Ex. 3. Let us inquire in Example 1, what will be the pressure sustained at the point B.

Let R be the pressure, and ϕ the angle which its direction makes with BD; also let $ACB = \theta$, $ADB = \theta'$; then we must have,

$$R\cos\phi - P\cos\theta + P'\cos\theta' = 0, \qquad (1)$$

$$R\sin\phi - P\sin\theta - P'\sin\theta' = 0, \qquad (2)$$

these equations correspond to (1) and (2) of Art. 17. The third equation of that Article, or the *equation of moments*, we have already used in Ex. 1; we will however repeat it, making use of our present notation; it will be as follows,

$$Pp\cos\theta - P'p\cos\theta' = 0, \text{ (since } BE = p\cos\theta,\ BF = p\cos\theta'\text{)},$$

$$\text{or } P\cos\theta - P'\cos\theta' = 0, \qquad (3)$$

equation (3) reduces (1) to the following

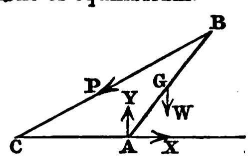
$$R\cos\phi = 0;$$

$$\therefore \phi = 90^0,$$

$$\text{and then } R = P\sin\theta + P'\sin\theta', \text{ from (2)}.$$

Hence therefore the pressure at B will be a *vertical* pressure, and equal to the sum of the vertical resolved parts of P and P'. This is a conclusion which might have been anticipated; but it is desirable to see how the result arises from the general equations of equilibrium.

Ex. 4. AB is a heavy beam, moveable in a vertical plane about A, and inclined to the horizon at an angle of $45°$; required the force which must be exerted by a man standing at C, where $AC = AB$, to prevent the beam from falling.



Let the weight of the beam be W; this we may regard as a single vertical force acting at the centre of gravity G of the beam, and G will be the middle point of AB if we regard the beam as uniform. Let P be the force required; and $AB = a$.

Then for equilibrium the moments of P and W about A must be equal;

$$\therefore\ P\times a\sin ABC = W\times\frac{a}{2}\cos 45^0,$$

$$\text{or } P\sin 22^\circ\ 30' = \frac{W}{2}\sin 45^\circ = W\sin 22^\circ\ 30'\cos 22^\circ\ 30';$$

$$\therefore\ P = W\cos 22^\circ\ 30',$$

$$= W\sqrt{\frac{1+\cos 45^0}{2}} = W\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}},$$

which is the force required.

Ex. 5. In the preceding example, as in Ex. 3, the two equations of equilibrium which we have not used will give us the pressure sustained by the ground at the point A. Instead of calling this pressure R as in the former instance, and denoting by ϕ the angle which its line of action makes with the horizon, we will take X and Y to represent its horizontal and its vertical resolved part respectively. We shall then have, if we resolve horizontally and vertically,

$$X - P\cos BCA = 0,$$

$$Y - W - P\sin BCA = 0;$$

$$\text{but } BCA = 22^\circ\ 30', \text{ and } P = W\cos 22^\circ\ 30';$$

$$\therefore\ X = P\cos 22^\circ\ 30' = W\cos^2 22^\circ\ 30' = \frac{W}{2}(1+\cos 45^0)$$

$$= \frac{W}{2}\left(1+\frac{1}{\sqrt{2}}\right),$$

$$\text{and } Y = W + P\sin 22^\circ\ 30' = W(1+\sin 22^0\ 30'\cos 22^0\ 30')$$

$$= W\left(1+\frac{\sin 45^0}{2}\right) = W\left(1+\frac{1}{2\sqrt{2}}\right).$$

These two expressions for the resolved parts X and Y entirely determine the magnitude and direction of the pressure; for if R and ϕ have the meanings above assigned to them, we have

$$R^2 = X^2 + Y^2, \text{ and } \tan\phi = \frac{Y}{X}.$$

Ex. 6. The general principles of equilibrium require, that the forces resolved in *any* two directions at right angles to each other should vanish, and that the moment of the forces about *any point* should also vanish. We will illustrate this by resolving the forces in the preceding

example in the direction of AB and perpendicular to it, and by taking the moments about B. In considering the problem thus we must regard the beam AB as under the action of the four forces P, W, X and Y; and we will slightly vary the problem by supposing AB to make with the horizon a given angle θ; then we shall have

$$W \sin\theta + P\cos\frac{\theta}{2} - X\cos\theta - Y\sin\theta = 0, \quad (1)$$

$$W\cos\theta - P\sin\frac{\theta}{2} + X\sin\theta - Y\cos\theta = 0, \quad (2)$$

$$Xa\sin\theta + W\frac{a}{2}\cos\theta - Ya\cos\theta = 0. \quad (3)$$

Multiply (1) by $\cos\theta$ and (2) by $\sin\theta$, and by subtraction there results,

$$P\left(\cos\theta\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2}\right) - X = 0,$$

$$\text{or } X = P\cos\frac{\theta}{2}.$$

Again, multiply (1) by $\sin\theta$, and (2) by $\cos\theta$, and by addition we have,

$$W + P\left(\sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2}\right) - Y = 0,$$

$$\text{or } Y = W + P\sin\frac{\theta}{2};$$

therefore from (3)

$$\frac{W\cos\theta}{2} = Y\cos\theta - X\sin\theta,$$

$$= W\cos\theta + P\left(\sin\frac{\theta}{2}\cos\theta - \sin\theta\cos\frac{\theta}{2}\right).$$

$$= W\cos\theta - P\sin\frac{\theta}{2};$$

$$\therefore\ P = W\frac{\cos\theta}{2\sin\frac{\theta}{2}},$$

$$X = W\frac{\cos\theta}{2\tan\frac{\theta}{2}},$$

$$Y = W + \frac{W\cos\theta}{2}.$$

If $\theta = 45^0$, these results agree with those already obtained.

CONVERSATION UPON THE PRECEDING CHAPTER.

P. I observe that in this chapter the principle of the lever is deduced from the parallelogram of forces; is it not capable of independent demonstration?

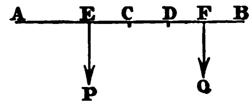
T. It is; and if the principle be so demonstrated, the parallelogram of forces may be deduced from it, and so a complete system of Statics may be built up, depending upon the principle of the lever as its foundation. It will be interesting to you to see how this can be done, the more so because the doctrine of the lever was the first mechanical doctrine which was placed upon a demonstrative basis, having been proved by Archimedes two centuries before the Christian æra.

The demonstration depends upon the following axiom. *Two equal weights* W, W, *supposed to be connected by a rigid rod without weight will balance upon the middle point of the rod and will produce there a pressure equal to* 2W.

This axiom you will find no difficulty in admitting, because, the weights being equal, there is no reason why one of them should descend rather than the other; and moreover, if the middle point of the rod be supported, the supporting point sustains the two weights, and therefore the pressure upon the point must be measured by the sum of the weights.

Hence it follows, that a uniform rod or cylinder will balance about its middle point, and will produce there a pressure equal to its weight; this is sometime expressed by saying that the statical effect of the rod or cylinder is the same as it would be if collected at its middle point. The truth of this immediately follows from the axiom just now enunciated, because we may consider the rod as cut up into any number of equal weights, and as each pair equidistant from the centre may be collected at the centre, the whole may be so collected.

Now let us take a uniform heavy rod AB, the weight of which is $P + Q$. This rod will balance about its middle point C.



Divide AB in D, so that

$$AD : DB :: P : Q,$$

then the weight of the portion AD is P, and that of DB is Q. Let E be the middle point of AD, and F of DB; then the statical effect of the rod AD is the same as that of a weight P suspended from E, and that of DB as that of a weight Q suspended from F. Hence the weights P and Q suspended from E and F respectively, will balance about C; and we have now only to determine by geometry what is the relation of the two arms CE and CF.

We have $CE = AC - AE = BC - ED = DB - CE$,

$$\therefore DB = 2CE;$$

$$\text{similarly, } AD = 2CF;$$

$$\text{but } P : Q :: AD : DB, \text{ by construction,}$$

$$\therefore P : Q :: CF : CE;$$

$$\text{or } P . CE = Q . CF.$$

That is, the moments of P and Q about C must be equal; which is the principle of the lever.

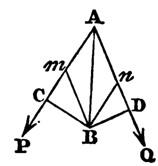
The proposition thus proved for two weights is true for any two parallel forces, and we can easily deduce the case in which the forces are not parallel.

P. I think you have already shewn me the method of doing this.

T. I have, in our conversation on the principle of the lever, treated experimentally. The method by which I then deduced (p. 44) the case of oblique forces acting upon a lever from that of two weights, will serve to deduce the same from Archimedes' proof of the proposition for parallel forces. We may therefore consider the proof of

Archimedes as a demonstration of the principle of the lever in its most general form; and starting from such a demonstration, I will now shew you how we can deduce the parallelogram of forces, and thus construct a complete system of mechanics.

Let Am, An represent in magnitude and direction two forces P and Q acting at the point A: complete the parallelogram $AmBn$, and draw AB. Also draw BC, BD, perpendicular to Am, An produced. Now suppose AB to be a rigid rod or lever, moveable about B, and acted upon by the forces P and Q at A. Then



$$\frac{P}{Q} = \frac{Am}{An} = \frac{\sin mBA}{\sin mAB} = \frac{\sin nAB}{\sin mAB} = \frac{BD}{BC},$$

$$\text{or } P \,.\, BC = Q \,.\, BD;$$

therefore the forces P and Q would keep the lever at rest.

And since the resultant of P and Q would produce the same effect as P and Q together, it also acting at A would keep the lever at rest. But no single force acting at A can keep the lever at rest unless it act in the direction AB, in which case it will only produce a pressure upon B which we suppose to be fixed; hence AB is the *direction* of the resultant of P and Q.

Having thus proved the parallelogram of forces as regards *direction*, it may be extended to the *magnitude* precisely as in Art. 4, p. 74.

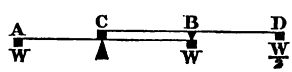
P. In this manner then all other propositions are reduced to that of two parallel forces upon a straight lever. Might we not in this way complete our system without reference to the theory of couples?

T. Doubtless: but the theory of couples puts the whole conception of the action of forces upon a rigid

body in the simplest and best point of view. The theory is due to the French mathematician Poinsot, the chief characteristic of whose mind seems to have been a remarkable power of presenting known truths in such new lights as to give them a simplicity of which we should scarcely have supposed them susceptible. The excellence of the theory of couples seems to be this, that it separates the effect of forces upon a rigid body into these two which are perfectly distinct, namely, the effect of *translation* and the effect of *twisting*. By the theory of couples we reduce all forces acting upon a rigid body to a force and a couple; if the former of these exist without the latter the body will not twist, and if the latter exist without the former it will twist only: if the body is to be at perfect rest both the one and the other must have zero for its value. It is perhaps not easy for you to realize the beauty of the Theory of Couples, so long as you are engaged only with the case of forces whose lines of action all lie in one plane; nevertheless it is desirable, that even in this early stage of your reading you should become familiar with a conception which occupies so conspicuous a place in modern Mechanics.

P. I suppose that there are other independent proofs of the principle of the lever besides that of Archimedes?

T. There are; but perhaps none which possess the same degree of simplicity. I will, however, take this opportunity of giving you another demonstration, which though not so complete, appears worthy of notice, because it exhibits the truth of the proposition in one case in a very easy and clear manner. The axiom upon which it rests is the same as before, namely, that two equal weights suspended from the extremities of a rod without weight will be in equilibrium, and produce at the middle point a pressure equal to their sum, but the application of the axiom is different. Let W, W be two equal weights balancing at the extremities of a horizontal rod AB about a fixed axis or fulcrum at its middle point C.



Let the weight W at B be replaced by two weights $\frac{W}{2}$, $\frac{W}{2}$ at the extremities of a rod CD, where $BD = BC$, the middle point of CD resting upon B; then the pressure at B is W as before, and the equilibrium is not disturbed. But the weight $\frac{W}{2}$ resting on C, which is fixed, cannot tend to turn the lever about C, hence the weight $\frac{W}{2}$ at D balances W at A, that is, *if we double the arm we must bisect the weight*, and this is the principle of the lever.

P. But only a particular case of it.

T. You are right; and it will not be worth while to attempt to complete the proof, that is, to shew that in *whatever* proportion we increase the arm we must diminish the weight; but the demonstration in the preceding case, which is extremely simple, exhibits the truth in that case so very clearly that it may be useful to you in assisting you to grasp the general truth. You will always find great advantage in viewing the same proposition in a variety of lights; of course if our only aim were to ascertain the truth for practical purposes, an experimental proof would answer our end as well as any other, but taking a higher view of the objects of science we ought not to allow ourselves to be satisfied with such a method, but should rather endeavour to ascertain, as far as may be, why propositions are true; and to this end varied methods of demonstration will assist us, and sometimes propositions which at first seemed difficult to prove will by continued attention at length appear so simple that they will hardly seem to need demonstration at all.

P. I believe I have now a tolerably clear perception of the principle of the lever, and of its application to the conditions of equilibrium of *rigid bodies;* but how am I to apply the principles of statics to bodies which are not rigid?

T. By a *rigid body* we mean, strictly speaking, a body which is incapable of bending or in any way changing its form, whatever may be the intensity of the forces which act upon it. To this definition however nothing in nature exactly corresponds: substances which in common language we should call very hard, as glass, steel, oak, are nevertheless by no means *rigid* in the mathematical sense of the word. But if a substance have such a degree of hardness, that its form is not sensibly changed by the action of the forces with which we are concerned, then we may apply without practical error the principles which have been proved for rigid bodies. For example, a steel bar of one inch in diameter and one foot long would not bend sensibly if an ounce weight were suspended from each end, therefore it might, under such circumstances, be treated as a rigid body. Of course there are many important cases of equilibrium to which our three statical equations will not apply; but without entering upon this more difficult ground we shall find abundance of problems of which those equations will afford the solution.

P. I perceive that we have in this chapter the algebraical sign − again introduced, as applied to couples. I am much struck with the application of this symbol to lines, angles, forces, and now to couples.

T. The employment of the two symbols + and −, as indicating contrary properties or qualities, is of admirable utility, and I think that the conception of it does not present any great difficulty; you will find rather different views given in different books of the principle upon which the use is founded, but I am not anxious to lead you into difficult speculations upon such points; when you become familiar with the application of the method, you will be in a better condition to form an opinion concerning the real ground upon which it is based.

P. I see no difficulty in the method, which can prevent me from using it with perfect confidence, and I believe

that I quite understand the principle, that for equilibrium the algebraical sum of the moments of the forces about any point must be zero, for this is equivalent to saying that there must be no tendency to twist about any point; but I am not sure that I understand how to determine the proper algebraical signs of the moments which occur in problems.

T. With a little practice you will be able to write them down without a mistake; the best instructions which I can give you are these. First, determine the point about which you will estimate the moments; in theory it is quite indifferent what point you take, but in practice one point may be more convenient than another; for instance, in Ex. 1 (p. 103), the post AB is kept in equilibrium by the three forces P, P', and R; but no question being asked about R, moments were taken about B, the point at which R acts, the consequence of which was that the moments of P and P' were the only moments which there was need to consider.

P. Why so?

T. Because no force can produce any moment, any tendency to twist, about the point at which it acts; taking our mathematical definition of moment we should say, that the perpendicular from the point of application of a force upon the direction of that force is zero, and therefore the moment also zero. On this account if there be any fixed point in the body, we should always take moments with reference to that point; if no point be fixed, it may frequently be a matter of indifference what point is chosen, but if there be one through which the directions of two or more forces pass, we should probably choose that as being most likely to give us the equation under the simplest form.

Having chosen the point about which moments are to be estimated, write down the moment of any one force; if the figure for the problem be properly constructed, the

forces being all represented in it with arrows denoting their directions, the eye will at once determine whether any particular force tends to twist the body in the same direction as that the moment of which has been written down, or in the opposite; if the former, write down its moment with the sign +; if the latter, with the sign −; and equate the whole to zero.

P. In Ex. 3. p. 104, it is stated that the result concerning the direction and magnitude of the pressure R might have been anticipated: will you explain this?

T. In deducing the parallelogram of forces from the principle of the Lever (p. 109), you will remember that it was taken for granted, that two forces P and Q, acting at the extremity A of a rod AB moveable about B, could not keep it at rest unless their resultant was in the direction AB; now this is precisely the case under consideration: P, P' produce a resulting force which must be in the direction AB; this must be met by an equal and opposite force R, arising from the pressure on the ground; but AB is vertical, therefore the direction of R must be vertical. R can have of course no tendency to counteract any horizontal force, therefore the horizontal parts of P and P' must be equal and opposite, and the two vertical portions must be counteracted by R. These are the results of the equations given in Ex. 3, which might therefore, as we see, have been anticipated.

The same conclusions will at once arise from the principle, that in considering the equilibrium of a rigid body moments may be taken about *any* point. For let us take moments about the point A, and let x be the length of the perpendicular from A upon the direction of R; then neither P nor P' produces any moment about A, therefore our equation of moments is

$$Rx = 0;$$

$$\therefore\ x = 0,$$

or the direction of R passes through A. And it is a

general principle, which is frequently useful in the solution of problems, that when a rigid body is acted upon by three forces, the directions of these forces must pass through the same point: for let the directions of two of the forces meet as they must in a certain point N, then by what has been said the direction of the third force must also pass through N. You will find hereafter in the Chapter of Problems how this principle may be conveniently applied.

EXAMINATION UPON CHAPTER V.

1. From the parallelogram of forces deduce the principle of the lever, the forces not being parallel.

2. Deduce the truth of the principle when the forces are parallel.

3. Shew that the principle of the lever if proved for a rigid rod may be extended to the case of any rigid body.

4. Define a *couple*, the *arm* of a couple, the *moment* of a couple.

5. The effect of a couple cannot be counteracted by the action of any single force.

6. The effect of a couple is not altered by turning its arm about one extremity through any angle in the plane of the forces.

7. The effect of two couples, the arms of which have a common extremity, and which tend to twist in the same direction, is the same, provided the moments be equal.

8. Shew how to find the resultant of any number of couples having the same plane.

9. Any system of forces, the directions of which lie in one plane, acting upon a rigid body, may be reduced to a single force and a single couple.

10. Investigate the conditions of equilibrium of a rigid body, the directions of the forces which act upon it lying all in one plane.

11. Prove, without assuming the parallelogram of forces, that two weights will balance upon a straight lever if their moments about the fulcrum be equal.

12. Assuming the principle of the lever deduce the parallelogram of forces so far as the *direction* of the resultant is concerned.

13. If a man who can just lift 3 cwt., pull at a post as in Ex. 2, p. 103, by means of a rope twice as long as the post is high, find what horizontal force must be applied at its middle point to prevent it from falling.

14. Three weights are suspended from the angular points of an equilateral triangle which is fixed in a vertical plane with one of its sides making an angle of 45° with the horizon; find the moment of the weights with respect to the centre of the triangle.

15. Two uniform beams of equal transverse section are fixed together by the extremities, so as to make with each other a right angle, and suspended from their point of junction; if one beam be twice as long as the other, find the position of equilibrium.

CHAPTER VI.

ON MACHINES.

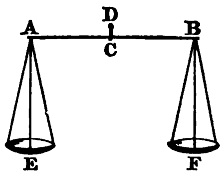
---

1. Any contrivance by means of which force is transmitted from one point to another, or by means of which force is modified with respect to direction or intensity, is called a *machine*. We have already had a simple instance of a machine in the case of the lever; the oar of a boat for example is a machine, for here the force applied at one end of the oar is converted into a force of propulsion at the rowlock; and in the same sense a poker, a crowbar, a pair of scissors, the human arm, may all be considered as machines. In this chapter we shall consider some other instances, and our object will be in each case to determine the conditions under which a certain force P, acting at one given point of a machine, will be in equilibrium with another force W, acting at another given point: P we shall usually call the *power*, and W the *weight*. Many machines are chiefly of practical use when they are in motion; thus in the case of the steam-engine the expansive force of steam is applied to put machinery in motion; but all calculations connected with machines in motion belong to the Science of Dynamics, not that of Statics, and we shall concern ourselves here only with examples of machines in equilibrium.

We shall begin by explaining two or three methods by which the property of the Lever is rendered available for the purpose of *weighing*.

2. *The Common Balance.*

Let AB be a rigid rod, CD a small rigid piece attached



to its middle point and perpendicular to it, and let *D* be supported by a string or otherwise. *E*, *F* are two scales or pans of equal weight suspended by strings from *A* and *B*. Then it is evident that if *A* and *B* be equally loaded, the *beam* *AB* will be horizontal, if not, that the more heavily loaded scale will cause the extremity to which it is attached to preponderate. And thus by placing any given weight, as 1 lb. for instance in the scale *E*, and putting such a quantity of any given substance into the scale *F* as shall allow of the beam *AB* being horizontal, we can weigh out a pound of that substance.

3. The preceding explanation represents the balance in its simplest form, and exhibits its principles: in practice many modifications and additional contrivances must be introduced; much skill has been expended upon the construction of balances, and great delicacy has been obtained. It would be beyond the scope of this book to describe all the features in the construction of firstrate balances, by means of which a degree of accuracy has been arrived at, which is truly wonderful: there are however two or three points to which it will be desirable to call attention.

The beam should be suspended by means of a knife-edge, that is, a projecting metallic edge transverse to its length, which rests upon a plate of agate or other hard substance. The chains which support the scales should be suspended from the extremities of the beam in the same manner.

The point of support of the beam should be at equal distances from the points of suspension of the scales; and when the balance is not loaded the beam should be horizontal.

To test the accuracy of a balance, first ascertain that the beam is horizontal when the balance is not loaded; then place two weights in the scales such that the beams shall be horizontal; lastly, change these weights into opposite scales, if the beam still remain horizontal the balance is a true one.

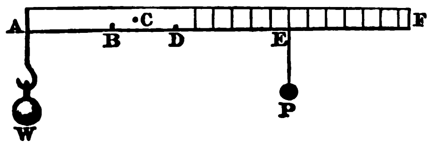
The chief requisite of a good balance is what is termed *sensibility;* that is to say, if two weights which are very nearly equal be placed in the scales, the beam should vary *sensibly* from its horizontal position. In order to produce this result two conditions should be satisfied; (1) the point of support of the beam and the points of suspension of the scales should be in the same straight line; the consequence of this will be that two equal weights in the scales will produce a resultant through the point of support, they will therefore have no effect whatever in twisting the beam, and the deviation from horizontality will be the same for a given *difference* of weights however great the weights themselves may be; (2) the point of support should be very near the centre of gravity of the beam, and a little above it; the nearer these two points are to each other the greater will be the sensibility, for the weight of the beam acting at its centre of gravity must be in equilibrium with the small difference of the weights acting at one end of the beam, and this difference of the weights will act at a greater mechanical advantage the nearer the centre of gravity of the beam is to the fulcrum.

If the sensibility of a balance be very great the addition of a small weight to either scale will cause the beam to oscillate, and some time will elapse before it attains its position of equilibrium; on this account the beam is sometimes furnished with a pointer and a graduated arc of a circle; if the pointer oscillates through equal arcs on opposite sides of the point which corresponds to horizontality, we may be satisfied that the scales are equally loaded, without waiting to ascertain whether the beam will ultimately rest in a horizontal position.

4. The common balance requires a series of weights in order to render it practically useful, but there is another kind of weighing machine in which one and the same weight is made use of in all cases. This is the instrument known as the Roman or Common Steelyard.

The Common Steelyard.

Let AF be a rigid bar moveable about a horizontal pivot at C; and from A let the weight W which we desire to measure be suspended. P is a given moveable weight, which can be suspended from any point E of the bar between C and F; and it is evident, from the principle of the lever, that the larger is W, the further must the point of suspension E be from C in order that the steelyard may be horizontal. Suppose then a certain weight suspended at A, the point of suspension of P must be shifted until the steelyard is horizontal, and the bar is so *graduated* that by looking at the number which is nearest to E we can at once ascertain the weight of W.



5. The process of graduating the steelyard deserves attention.

To graduate the common steelyard.

Remove the weights P and W, and suppose that under these circumstances the arm CF of the steelyard preponderates; find, by trial, the point B, such that if P be suspended from B the steelyard will be horizontal; take $CD = CB$, then the moment of the weight of the steelyard about C is the same as that of P suspended from D. Now let W hang from A, and P from any point E, then for equilibrium we must have

$$W \times AC = P \times CD + P \times CE = P \times BE;$$

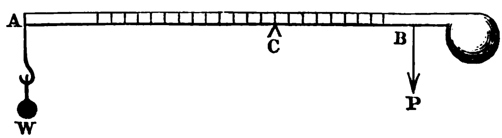
$$\therefore\ BE = \frac{W}{P} . AC.$$

Suppose that $P = 1$lb.; and make W successively $=$ 1lb., 2lbs., 3lbs., &c., then the values of BE will be AC, $2AC$, $3AC$, ..., and these distances must be set off, measuring from B, and the points so determined marked 1lb., 2lbs., 3lbs., &c.

6. Another form of this balance is that which is called the *Danish Steelyard*, in which the weight is fixed to the beam and the fulcrum is moveable. This is, for the greater number of purposes, not so convenient a construction as the preceding; it is however not inconvenient for weighing small weights, when no great accuracy is required; letter-balances are sometimes made upon this principle.

To graduate the Danish Steelyard.

Let B be the point on which the instrument would balance, if no weight were suspended at A; and when the weight W is suspended at A let C be the place of the fulcrum; also let P be the entire weight of the instrument, which may be supposed to be collected at B, or which, in other words, will produce a downward pressure at B equal to P. Then for equilibrium we must have



$$W \times AC = P \times BC = P(AB - AC);$$

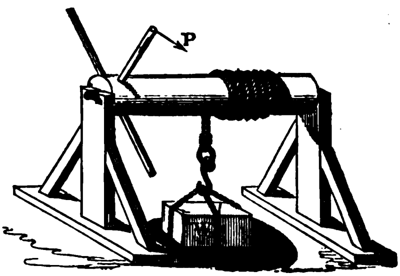
$$\therefore\ AC = \frac{P}{W+P} . AB.$$

Hence, making $W =$ 1lb., 2lbs., 3lbs. ... successively, we shall be able to mark upon the steelyard the corresponding positions of the fulcrum; and when the beam is thus graduated we shall be able to ascertain the weight of any given body suspended from A, by observing the mark of graduation which is nearest to the fulcrum.

7. It will be seen that the distances between the successive marks of graduation on the common steelyard are equal, but on the Danish unequal. In fact, the distances of the successive marks of graduation from A, the extremity of the beam which supports W, in the common steelyard form an *arithmetical* progression, in the Danish they form an *harmonical.*

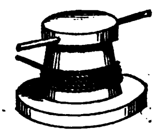
8. The principle of the lever may be conveniently applied for the purpose of lifting or sustaining great weights; this is done by means of a *windlass* or *capstan*.

The *windlass* is used for such purposes as that of raising an anchor. It may be described as a strong cylindrical beam, moveable about a horizontal axis, the extremities being inserted into two strong upright pieces in which they are capable of turning freely. One end of a rope is coiled partially



round the windlass, and to the other end is attached the anchor or the weight to be raised; a number of apertures are made in the windlass perpendicular to its axis, and in these are inserted short bars called *handspikes;* by means of these it is evident that the windlass may be made to revolve, and when by its revolution a handspike is brought inconveniently low it is taken out and reinserted in a more convenient place. The windlass in the figure is represented with fixed bars, instead of handspikes, which in some applications of the machine is a more convenient arrangement.

9. Some inconvenience arises from the necessity of changing the position of the handspikes; this is avoided in the *capstan,* the principle of which is the same as that of the windlass, but the axis is vertical, and a person may therefore by moving his own position cause the capstan to revolve without changing the point of insertion of the handspike.

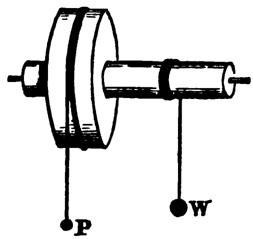


10. In both the preceding cases the mechanical advantage gained depends of course upon the length of the handspike, which however is limited by considerations of practical convenience. The actual relation between the *power* and *weight* upon machines of this kind will be seen

in the investigation of these conditions for the machine known as

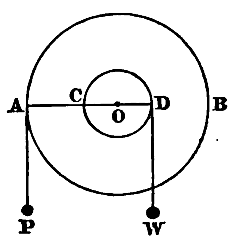
The Wheel and Axle.

This machine consists, in its simplest form, of two cylinders having their axes coincident; the two cylinders forming one rigid piece; the larger is called the *wheel*, the smaller the *axle*. The cord by which the weight is suspended is fastened to the axle and coiled round it; the power may be supposed to act in like manner by means of a cord coiled round the wheel, as in the figure; or the power may act by means of a handle, as in the case of the common well and bucket.



11. *To find the ratio of* P *to* W, *when there is equilibrium upon the Wheel and Axle.*

Let AB, CD represent sections of the wheel and axle respectively, and O their common centre; P and W the power and weight, acting by means of strings at the circumference of the wheel and axle respectively.



For simplicity's sake P, W, and the arms at which they act, are in the figure represented in the same plane.

From the common centre O draw OA, OD to the points at which the cords supporting P and W touch the circumferences of the wheel and axle respectively; these lines will be perpendicular to the directions in which P and W act; hence, by the principle of the lever, or in other words taking moments about O,

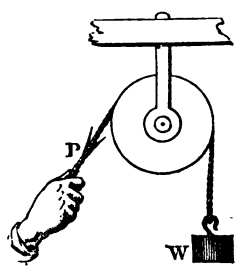
$$P \times AO = W \times OD,$$

$$\text{or } \frac{P}{W} = \frac{OD}{AO} = \frac{\text{radius of axle}}{\text{radius of wheel}}.$$

It is evident that the larger the radius of the wheel, the greater will be the mechanical advantage, that is, the smaller will be the power P necessary to support or to raise any given weight W.

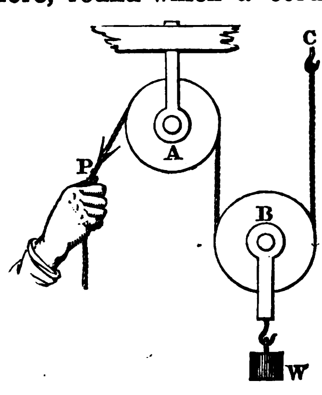
12. *The Pully.*

The Pully, in its simplest form, consists of a wheel, capable of turning about its axis, which may be either fixed or moveable. A cord passes over a portion of the circumference; if the axis of the pully be fixed, the only effect of the pully is to change the direction of the force exerted by the cord, and in this case no mechanical advantage is gained so far as the intensity of the force is concerned. Nevertheless the contrivance may be very convenient; for example, if we wish to raise a heavy weight, we can frequently do so most conveniently by attaching to it a cord which passes over a fixed pully, as in the figure; the effort which must be exerted in this case to raise the weight is the same as that which would be exerted to raise it without the intervention of the pully.



But suppose we modify the preceding contrivance as follows.

Let A be a fixed pully as before, round which a cord passes, and let this cord, instead of being made fast to the weight W, pass round a moveable pully B from which the weight depends, and then be made fast to a fixed point C. In this case, not only is the direction of the force changed, so that a person pulling *downwards* raises the weight, but also the force which he will have to exert will be equivalent to only half the weight

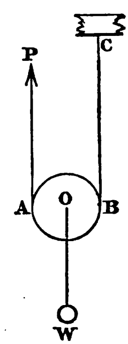


raised; for instance, a weight of 1lb. suspended at the *power end* of the cord will raise a weight of 2lbs.; and we shall find that by various combinations of pullies still greater advantage can be gained; in fact, by a sufficiently complicated system of pullies we can make a given force support any weight however large. We shall investigate the relation of P to W in the case of the single moveable pully, and also in the case of several complicated systems; these systems may be multiplied to any extent, but the method of finding the relation of P to W will apply *mutatis mutandis* to all.

In practice the pullies are made of wood or metal, and are therefore heavy bodies, whose weight ought in strictness to be taken into account; but for simplicity's sake we shall neglect the weight of the pullies, as for like reasons we shall that of the cord which passes round them. We shall also suppose the portions of cord to be parallel and vertical.

13. *To find the ratio of the Power to the Weight in the single moveable Pully.*

Let O be the centre of the pully, which is supported by a cord passing under it and attached to a fixed point C at one end, and stretched by the force P at the other. Suppose the weight to be suspended from the centre O.



Then the pully with its depending weight W is supported by two strings AP and BC; the tension of the former is P, because by hypothesis the force P acts at the end of it; that is to say, the string AP exercises a supporting force upon the pully equal to P. Now the string BC which acts upon the other side of the pully is similarly circumstanced to AP, and must therefore exert an equal supporting force upon the pully. Hence on the whole the pully is acted upon by two equal forces,

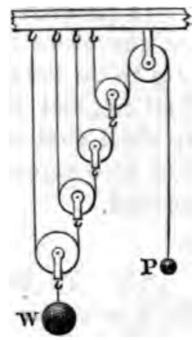
each equal to P, upwards, and by the weight W downwards, and therefore we must have

$$2P = W,$$

$$\text{or, } \frac{P}{W} = \frac{1}{2}.$$

14. *To find the ratio of the Power to the Weight in a system of Pullies, in which each pully hangs by a separate string.*

This system is represented in the figure, and is usually spoken of as the *First System of Pullies.*



By the property of the single pully the tension of the string which supports the lowest pully will be $\frac{W}{2}$. The tension of the string which supports the lowest but one will be $\frac{W}{2^2}$, and so on. Let there be n pullies, n being any number; then the tension of the string which supports the n^{th} pully will be $\frac{W}{2^n}$; but this must be equal to P, since the tension of the string which supports the n^{th} pully, is produced by the force P;

$$\therefore\ P = \frac{W}{2^n}, \text{ or } \frac{P}{W} = \frac{1}{2^n}.$$

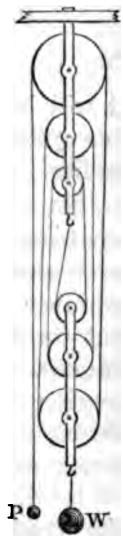
It will be seen, that in this system the mechanical advantage gained increases very rapidly with the number of pullies; thus if

$n = 2$, a weight of 1 lb. will support 4 lbs.,
$n = 3$, ... 8 lbs.,
$n = 4$, ... 16 lbs.,

and so on.

15. *To find the ratio of the Power to the Weight, in a system of Pullies in which the same string passes round all the Pullies.*

This system will be understood from the figure, and is known as the *Second System of Pullies.*



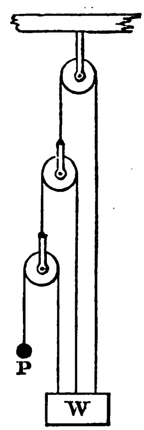
There are two blocks, the lower one moveable, the upper one fixed, and each containing a number of pullies. The same string goes round all the pullies, and therefore the tension throughout will be the same, and equal to the power P. Let n be the number of strings at the lower block, then the sum of their tensions will be nP, and we shall have

$$nP = W, \quad \text{or} \quad \frac{P}{W} = \frac{1}{n}.$$

The mechanical advantage does not, in this system, increase so rapidly with the increase of the number of pullies as in the previous system; but on many accounts it is practically more convenient.

16. *To find the ratio of the Power to the Weight in a system of Pullies in which all the strings are attached to the weight.*

This system is represented in the figure, and is known as the *Third System of Pullies.*



The tension of the string which supports P is P; that of the next string is $2P$, by the property of the single pully; that of the next is 2^2P; and so on. Let there be n strings, then the tension of the last is $2^{n-1}P$; and the sum of all the tensions is

$$(1 + 2 + 2^2 + \ldots + 2^{n-1})P, \text{ or } (2^n - 1)P.$$

But the sum of all the tensions must be equal to W, since the strings support W;

$$\therefore\ (2^n - 1)P = W, \quad \text{or} \quad \frac{P}{W} = \frac{1}{2^n - 1}.$$

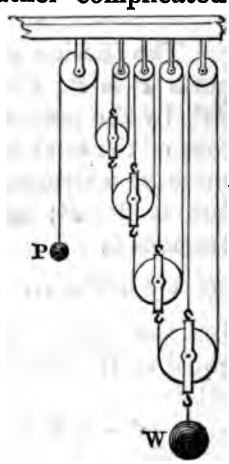
For instance,

if $n = 2$, a weight of 1lb. will support 3lbs.
$n = 3$, ... 7lbs.
$n = 4$, ... 15lbs.

It will be seen, that the gain of mechanical advantage in this system is nearly the same as in the first system of pullies.

17. The principles upon which the relation of P to W has been determined in the preceding articles are (as has been already remarked) applicable to all systems of pullies, however complicated. A rule may be given, as follows, but its meaning will be best seen by applying it to examples. Begin at the *Power-end* of the system, then the tension of the string which supports P will be equal to P throughout; against each of the parallel portions of this string write P; now proceed to the next string, find what its tension is by observing how many strings, each having the tension P, produce it; write the expression for its tension against each parallel portion of it; and so with the next string. When the tension of each string of the system has been written down, it is easy to see how many of them support W, and by adding their tensions together we have the relation between P and W required.

18. We will illustrate this by a rather complicated system, represented in the figure. The P string occurs three times, and produces a tension $3P$ in the next string; this again occurs three times, and therefore produces a tension 3^2P or $9P$ in the next; and so on. If we have three pullies, as in the figure, the result will be



$$27P = W, \text{ or } \frac{P}{W} = \frac{1}{27}.$$

If more generally we take n pullies, we have

$$3^n P = W, \text{ or } \frac{P}{W} = \frac{1}{3^n}.$$

19. *The Inclined Plane.*

By an inclined plane is meant a plane inclined to the plane of the horizon, and the angle which it makes with the plane of the horizon is called the *inclination* of the plane.

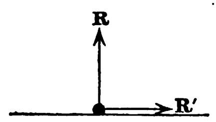
If a weight be placed upon a *horizontal* plane it will rest in the position in which we place it, because the effect of a body's weight is in this case only to make it press against the plane, which returns the pressure; but if we place a weight upon a smooth *inclined* plane, unless it be supported, it will slide down, for, in this case, the tendency which the body has to descend is not entirely checked by the plane. In practice, a body will remain at rest upon the surface of a plane of considerable inclination, but this arises from the fact that in practice all bodies are more or less *rough*, and the roughness of the inclined plane will be sufficient to prevent a weight from sliding down it, if the inclination be not very great; we shall say something more upon this subject when we come to the general consideration of *friction;* at present we shall suppose that the inclined plane is perfectly smooth, that is, that it is incapable of offering any resistance to the sliding of a body along its surface.

The problem in the case of the inclined plane is this, to determine what force P, acting in a given direction, will support a given weight W, resting upon a plane of given inclination. It may perhaps be asked, how this problem properly comes under the head of *machines;* but it will be seen by reference to our definition of a machine in Art. 1. (p. 117) that the inclined plane is rightly so regarded, for it supplies us with the means of modifying the effects of a given force. Moreover, an example will shew that the inclined plane may be used as a means of assisting human strength, in the same manner as the lever or the pully: for let it be required to raise a cask of wine from a cellar, then we may either roll the cask to the side of the cellar, and extract it by means of a crane and pully, or we may lay down

some planks at a moderate inclination and drag up the cask upon them.

20. Before we proceed to find the relation of P to W upon the inclined plane, we must make an important remark respecting the pressure exerted by a plane upon a body which rests upon it. If a particle rests upon a horizontal plane the forces which keep it at rest are *two;* viz. the weight of the particle *downwards*, and a certain pressure caused by the plane *upwards*, and these must be equal, otherwise the particle could not be at rest; hence in considering the equilibrium of such a particle we may dismiss all thought of the plane, and say that the particle is kept at rest by its own weight W acting vertically downwards, and a pressure W acting vertically upwards. Now let us consider what will be the mechanical effect of a smooth plane, which is not horizontal, upon a particle made to rest upon it. Its effect will be to produce a pressure upon the particle; and there will be two questions, what will be the direction of this pressure, and what will be its magnitude?

(1) For the direction, we can at once conclude that the pressure must be perpendicular to the plane; because the plane is by hypothesis *smooth*, and by the term *smooth* we mean that it is incapable of offering any resistance to the motion of a particle along its surface. To make this more clear, suppose the pressure exerted by the plane to be in any direction whatever: then since a force may always be *resolved* into two at right angles to each other, let this pressure be resolved into two, one perpendicular to the plane, which call R, and one parallel to the plane, which call R': now the force R' will manifestly tend to make the particle move along the surface of the plane; but this is contrary to the definition of a *smooth* plane, therefore $R' = 0$; and hence the only force exerted by the plane is a force R in the direction perpendicular to it. But



(2) What will be the magnitude of R? This we have not sufficient data to determine; it will vary in different

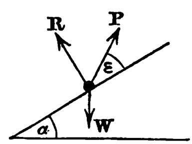
cases, and we cannot determine it until we have all the circumstances given.

Hence in any given problem we may consider the effect of a smooth plane to be this, *to produce a force or pressure upon a particle in contact with it, in the direction perpendicular to it, but of unknown magnitude, and which we must therefore denote by a symbol for an unknown quantity such as R.*

This being premised, we proceed,

21. *To find the ratio of the Power to the Weight, when there is equilibrium on the Inclined Plane.*

Let α be the inclination of the inclined plane to the horizon; R the pressure of the plane on the weight W, which pressure will be perpendicular to the plane; and, to take the most general case, let the direction of the power P make an angle ϵ with the plane.



Then resolving the forces parallel and perpendicular to the plane, we have

$$P \cos \epsilon - W \sin \alpha = 0 \text{ .... } (1),$$

$$R + P \sin \epsilon - W \cos \alpha = 0 \text{ .... } (2).$$

$$\text{Hence, } \frac{P}{W} = \frac{\sin \alpha}{\cos \epsilon}, \text{ from } (1).$$

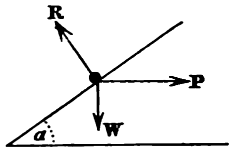
Equation (2) gives us the pressure upon the plane; thus

$$\begin{aligned} R &= W \cos \alpha - P \sin \epsilon, \\ &= W \cos \alpha - W \frac{\sin \alpha}{\cos \epsilon} . \sin \epsilon, \\ &= \frac{W}{\cos \epsilon} (\cos \alpha \cos \epsilon - \sin \alpha \sin \epsilon), \\ &= W \frac{\cos (\alpha + \epsilon)}{\cos \epsilon}. \end{aligned}$$

There are two particular cases, which are worthy of notice.

(1) Suppose the power acts parallel to the plane, then the equations become

$$P - W \sin \alpha = 0,$$
$$R - W \cos \alpha = 0;$$
$$\therefore \frac{P}{W} = \sin \alpha, \text{ and } \frac{R}{W} = \cos \alpha.$$



(2) Suppose the power acts horizontally; then the equations will be,

$$P \cos \alpha - W \sin \alpha = 0,$$
$$R - P \sin \alpha - W \cos \alpha = 0;$$
$$\therefore \frac{P}{W} = \tan \alpha,$$

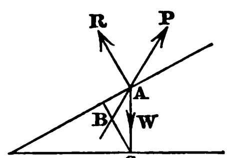
$$\text{and } R = W \cos \alpha + W \frac{\sin^2 \alpha}{\cos \alpha} = W \frac{\cos^2 \alpha + \sin^2 \alpha}{\cos \alpha} = \frac{W}{\cos \alpha}.$$

It may be remarked that these results may be deduced from those of the general case by making

$$\epsilon = 0 \text{ and } \epsilon = -\alpha.$$

22. For the sake of illustration we will solve the problem of the inclined plane in another way.

Let α, ϵ, R represent the same quantities as before. Let A be the point of the plane at which the weight rests; draw AC vertical, and from C draw CB in a direction perpendicular to the inclined plane, to meet the line of P's action in B. Then the sides of the triangle ABC, being parallel to the directions of the forces P, R, W, may be taken to represent these forces (Art. 6, p. 74). Hence



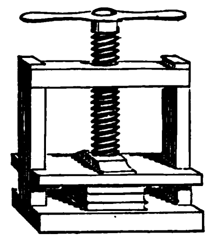
$$\frac{P}{\sin C} = \frac{W}{\sin B} = \frac{R}{\sin A}.$$

But $A = 90^0 - \alpha - \epsilon$, $B = 90^0 + \epsilon$, $C = \alpha$;

$$\therefore \frac{P}{\sin \alpha} = \frac{W}{\cos \epsilon} = \frac{R}{\cos (\alpha + \epsilon)}.$$

These are the same results as those obtained in Art. 21.

23. The last machine which we shall consider is *the Screw.* This machine in combination with the lever is of great practical utility, as, for instance, in the case of a bookbinder's press, in which a considerable pressure is required, and may be by this means produced with great facility; and there are numberless other examples.



The Screw may be described as an inclined plane wrapped round a cylinder, or as a cylinder having on its surface a projecting thread in all parts at the same given angle to the horizon: take any solid body of a cylindrical form, as, for instance, a ruler, a pencil; take a piece of paper ABC in the form of a right-angled triangle, having the right-angle at C; place BC upon the cylinder, so as to be parallel to its axis, and wrap the paper closely upon the cylinder, then the hypothenuse AB will mark out the thread of a screw. The form of the thread is different in different cases; it may be such as in fig. I., or such as in fig. II.; but this is a matter into which we shall not enter, and we shall consider the thread only as the surface of an inclined plane wrapped round a cylinder as before described.

fig. I.

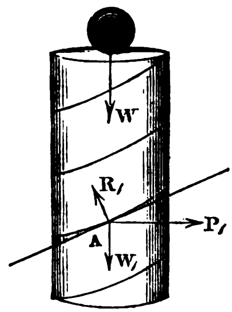
fig. II.

The screw is applied as follows: the cylinder bearing the thread fits into a block pierced with an equal cylindrical aperture, upon the inner surface of which is cut a groove, the exact counterpart of the thread of the screw; hence the screw can only be made to move in the block by revolving about its axis. Suppose the axis of the screw to be vertical, and a weight W to be placed upon it, then the screw would descend, unless prevented from doing so by another force; this force we will suppose to be supplied by the power P acting in a horizontal direc-

tion, at the extremity of an arm of given length. In practice there must necessarily be considerable friction between the thread and the groove, but this we shall not consider, because it would complicate the problem.

24. *To find the ratio of the Power to the Weight in the Screw.*

Let the power P act at an arm a, and let r be the radius of the cylinder, α the inclination of the thread to the horizon.



Consider the equilibrium of any point A of the thread; suppose the portion of the thread on each side of A to be unwrapped, so as to assume the position of a straight line BC, inclined at an angle α to the horizon; then we may consider the point A as supported upon a plane of inclination α, and acted upon by the pressure of the plane, which call R_1, a certain horizontal force caused by P, which call P_1, and a portion of the weight W, which call W_1; hence we shall have by resolving the forces in the direction BC,

$$W_1 \sin \alpha = P_1 \cos \alpha.$$

In like manner, if we considered the equilibrium of any other point of the thread, we should have

$$W_2 \sin \alpha = P_2 \cos \alpha;$$

and so on.

Hence, taking account of all the points of the thread, and adding all the equations together, we shall have

$$(W_1 + W_2 + W_3 + \ldots) \sin \alpha = (P_1 + P_2 + P_3 + \ldots) \cos \alpha.$$

But $W_1 + W_2 + W_3 + \ldots =$ the whole weight supported $= W$. Also, $P_1 + P_2 + P_3 + \ldots =$ the whole horizontal force supposed to act at the circumference of the cylinder, that is,

at an arm r. But the horizontal pressure is caused by P acting at an arm a; hence, by the principle of the lever,

$$(P_1 + P_2 + P_3 \ldots) r = Pa;$$

$$\therefore\ W \sin \alpha = \frac{Pa}{r} \cos \alpha,$$

$$\text{or } \frac{P}{W} = \frac{r}{a} \tan \alpha.$$

We may put this result in a more convenient form thus;

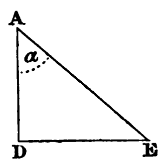
$$\frac{P}{W} = \frac{2\pi r \tan \alpha}{2\pi a},$$

$$= \frac{\text{the vertical distance between two threads}}{\text{circumference of circle described by } P}.$$

25. This investigation may be presented under a rather different form as follows:

The weight W may be conceived of as a body acted upon by these three forces, its own weight W vertically downwards, a certain force in a direction perpendicular to the thread of the screw, which will be the resultant of the pressures at the different points of the thread, and lastly, a horizontal force which we will call Q.

Now draw AD vertical, AE in a direction perpendicular to the thread of the screw, and DE horizontal; then the sides of the triangle ADE, being parallel to the three forces just now described, may be taken to represent them;



$$\therefore\ \frac{Q}{W} = \frac{DE}{AD} = \tan \alpha.$$

But Q is a force which at an arm r is in equilibrium with P at an arm a,

$$\therefore\ Q \times r = P \times a, \text{ or } Q = P\frac{a}{r},$$

$$\therefore\ \frac{P}{W} = \frac{r}{a} \tan \alpha,$$

which is the result previously obtained.

CONVERSATION UPON CHAPTER VI.

P. I feel somewhat disappointed with the contents of this chapter; I had expected from the title *Machines* that it would have treated of steam-engines and the like, and have introduced me to some of the wonders of modern machinery.

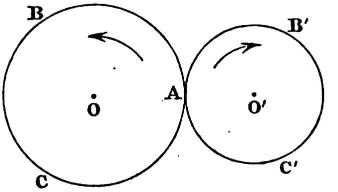
T. You must remember that there are two subjects very similar in name but very different in nature, namely, *Mechanics* and *Mechanism.* The science of Mechanism treats of the construction of complicated machines, such as steam-engines, looms, clocks, mills, and the like, and it describes the trains of machinery by means of which the moving power, whether it be that of steam, horses, weights, springs, or wind, is made to produce the required result. Scarcely any subject can be more interesting than this, and for an Englishman especially (England being the very land of machinery) a competent acquaintance with it would seem to be almost a necessary part of a good education; but you must not look for this description of machinery in a mathematical treatise upon Mechanics: the science of Mechanics is the science of Force; the laws of force in the case of Statics reduce themselves to the equations of equilibrium, and these can be investigated without any reference to machines, and those machines which we have considered have been introduced chiefly as illustrations of the principles of equilibrium. I should wish you to consider this chapter on Machines as merely subordinate to the general purpose of the book, not at all as the chief end of it: in a practical point of view, of course, it is the application of the principles of equilibrium to actual cases which constitutes their chief value; but regarding the study as a part of a liberal education, we should learn chiefly to estimate it because it is a portion of demonstrated truth; or if you wish for practical applications, rather look to the wonderful motions of the heavenly bodies, concerning which you may read in popular works, but which you

cannot possibly understand unless you have mastered the principles of the science of force.

P. But surely mechanics must form a considerable branch of mechanism.

T. It does, but perhaps not its characteristic branch. The chief problem with which the mechanician has to concern himself is the transmission of motion from one part of a machine to another, and the converting of one kind of motion with another, the conversion of rectilinear motion into circular for instance, or *vice versâ*. If, for illustration, you turn your eye to Professor Willis's *Principles of Mechanism*, you find such heads as these in the general Table of Contents, "Communication of Motion by Rolling Contact, by Sliding Contact, by Wrapping Connectors, by Linkwork." You will see at once from this that a book which undertakes to teach you the principle of the construction of modern machinery, is chiefly occupied with something quite different from the science of Force. And this gives me an occasion to observe, that I have omitted from the chapter on Machines one which is usually included, namely, the *toothed wheel*; I have done so because it involves no important mechanical principle, and is chiefly useful as a means of transmitting and modifying motion in machinery.

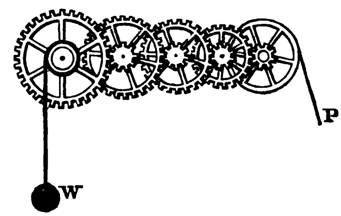
In order to understand the construction of toothed wheels let BAC, $B'AC'$ be two wheels, lying in the same plane, turning about centres O, O', and being in contact at A. And suppose that the friction between the surfaces of these two wheels is so great that



they cannot slide one upon the other; then if we turn the wheel BAC in the direction of the arrow marked upon it, it is evident that the wheel $B'AC'$ must also turn, but in the opposite direction, that is, in the direction indicated in the figure by the arrow upon $B'AC'$. Thus the motion of

BAC produces motion in *B'AC'*; and *B'AC'* may in like manner be made to *drive* another wheel, and so on. Moreover, it is easy to see that if the magnitude of *BAC* be given, and the wheel be made to turn at a given rate, the wheel *B'AC'* will turn more slowly or more rapidly than *BAC* in exact proportion as its circumference (or its diameter) is greater or less than that of *BAC*. For instance, suppose the diameter of *BAC* to be two feet, and that of *B'AC'* to be one foot; then if *BAC* be made to turn 30 times in a minute, *B'AC'* will turn 60 times, and so on. Hence wheels connected as I have described may be made both to transmit and also to modify motion.

But practically it is not possible to construct wheels the surfaces of which shall drive accurately by means of friction only; hence the device of *teeth*, that is of alternate projections and hollows upon the surface of the wheels. The figure will shew at once, after what I have said, the action of the teeth. Mechanically speaking a train of wheel-work is only a succession of levers, and if the number and magnitude of the wheels be given, there is no difficulty whatever in determining the relation between the power *P*, which, acting at the circumference of one extreme wheel of the train, will be in equilibrium with the weight *W* acting upon the other extreme wheel.



P. I have frequently observed a difference of shape in different teeth; is there any one which is better than another?

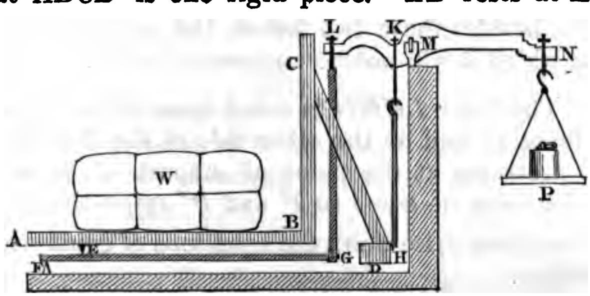
T. The subject of the proper form for the teeth of wheels is one of considerable complexity, and one upon which we cannot now enter. It is, however, of extreme practical importance, and I may mention that it is discussed very fully in Professor Willis's *Principles of Mechanism*. I ought to remark to you that if in considering the action of one wheel upon another we regard the

wheels as circles in contact with each other, each pair of wheels will exercise upon each other an equal and opposite action in the direction of their common tangent; but if we take account of the teeth, then the direction of the action between two teeth which happen at any moment to be in contact will be perpendicular to the common tangent of the surfaces of the teeth; this you will understand better, when you have read the next chapter.

P. There are many other weighing machines besides those described in this chapter: are there not?

T. Yes, there are; sometimes the extension or compression of a spring, sometimes the lifting of a weight at the end of a lever through a space ascertained by means of a graduated arc, is made use of for the construction of weighing machines. These, however, are omitted in the chapter which you have been reading, because they do not introduce any new and important mechanical principle; you will find descriptions of them in popular books, such as the treatise on Mechanics in Lardner's *Cabinet Cyclopedia*, and the like. Some of the weighing machines however, which are formed by a complicated system of levers, are worthy of your attention in a mechanical point of view; I will describe the construction of one as an example of this class, and we will work out the relation between P and W which belongs to it.

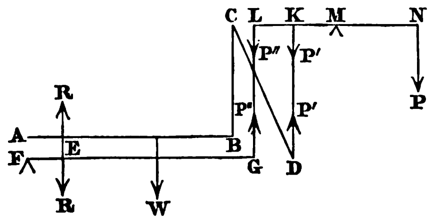
A platform AB, upon which the body which we wish to weigh is placed, is supported at one end by the piece BC which is connected (as shewn in the figure) with D, so that $ABCD$ is one rigid piece. AB rests at E upon



a lever FG, (concerning which more presently), and at the other extremity it is supported by means of a rod HK connected with the piece D. The lever FG, which has F for its fulcrum, is also supported by a rod GL; and both HK, and GL, are finally supported by a lever $LKMN$, which having M for its fulcrum, carries at its other extremity a scale in which is placed the weight P which is to be in equilibrium with W.

This is a general description of the machine in question; now let us reduce it to its simplest statical form.

The figure represents the statical problem; the letters in it correspond to those in the preceding description; and it will be seen that the machine consists essentially of three portions; the platform $ABCD$, the lever FG, and the lever $LKMN$; the two rods GL, DK only serve to connect these parts. Let us now consider the forces to which they are severally subject.



(1) The platform $ABCD$ is acted upon by the downward pressure of the weight W, and by two upward pressures at the two points of support, which (as we do not know their values) we will denote by R and P' respectively.

(2) The lever FG is acted upon by a downward force at E which must be equal and opposite to R, and by an upward force at G, which we will denote by P''. There is, of course, besides these two forces, the pressure upon the fulcrum, which it will not be necessary for us to consider.

(3) The lever $LKMN$ is acted upon at one extremity by the force P, and on the other side of the fulcrum there are two pressures at the point of support of the rod DK, GL, which must be equal to P' and P'' respectively.

We can now write down the equations of equilibrium for the machine.

For the platform $ABCD$ we must have for the equilibrium of the vertical forces,

$$W = R + P', \ldots\ldots\ldots\ldots\ldots\ldots (\alpha).$$

For the lever FG, taking moments about the fulcrum F,

$$R.FE = P''.FG \ldots\ldots\ldots\ldots\ldots\ldots (\beta).$$

For the lever $LKMN$, taking moments about the fulcrum M,

$$P.MN = P'.MK + P''.ML \ldots\ldots (\gamma).$$

If between (α) and (β) we eliminate the unknown quantity R, we have

$$W.FE = P'.FE + P''.FG \ldots\ldots (\delta).$$

We have now reduced the problem to the two equations (γ) and (δ), which you will observe involve two unknown forces P' and P''; and a little consideration will convince you that we have not omitted any equation essential to the solution of the problem.

P. The problem then is indeterminate.

T. As I have described the machine it is; but there is a peculiarity in the construction which I have omitted in my description, because you would not have then seen the necessity for it; the machine is so arranged that the following proportion holds amongst the lengths of the arms of the levers,

$$FE : FG :: MK : ML.$$

If we introduce this condition into equations (γ) and (δ), we have

$$\frac{P.MN}{MK} = P' + P''\frac{ML}{MK} = P' + P''\frac{FG}{FE} = W;$$

$$\therefore \frac{P}{W} = \frac{MK}{MN}.$$

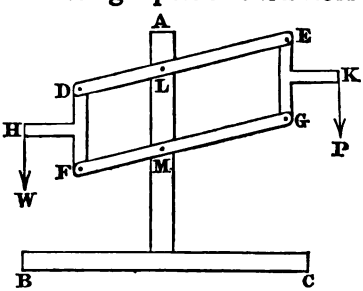
This is the required relation between P and W. You will now see the advantage of the machine: suppose, for instance, that $MN = 10 \times MK$, then $\frac{P}{W} = \frac{1}{10}$, or a weight of 10lbs. put into the P scale will serve to weigh 100lbs.

placed upon the platform. Machines upon this and similar constructions are frequent at railway stations, and are very convenient for weighing heavy goods. I should tell you that I have omitted some small points of adjustment which are necessary for accuracy, because I am anxious chiefly to draw your attention to the machine as a good illustration of statical principles.

P. It would seem from the investigation, to be indifferent at what point or points of the platform the pressure *W* takes place.

T. It is so; and the same kind of advantage belongs to the balance so extensively used in retail trades, and which is known as Roberval's balance. The construction of this balance is worthy of your notice; viewing it merely in its mechanical principles it may be described thus.

ABC is a firm vertical stand resting upon a fixed horizontal base; *DE, FG* are two equal bars working about pivots similarly situated with respect to their lengths, at *L* and *M*; *DFH, EGK* are two T-shaped pieces connected with the bars *DE, FG*, as in the figure, by pivots; the consequence of which is, that



if the system be made to assume different positions, *DF EG* will always be vertical. Now suppose two weights *P* and *W* suspended from *K* and *H* respectively; then in order to investigate the relation of *P* to *W* I might consider, as in the case of the weighing machine last described, the equilibrium of the separate members of the balance successively; but I think that without doing so I can make the result intelligible by general reasoning. Consider the weight *W*; it is supported by the piece *DFH*, which again is supported at *D* and *F*; now the effect of this must be to produce two vertical pressures at *D* and *F* which shall together be equal to *W*; it does

not signify to our purpose how much one point bears and how much the other, nor what horizontal pressure there may be at either, (since whatever horizontal pressure there may be at one of them, there must be an equal and opposite horizontal pressure at the other,) the pressure must on the whole be W, or we may say that there is a pressure W in the direction of the rod DF; so that as far as the equilibrium of the bar DE is concerned we may regard W as suspended from D; and in like manner we may regard P as suspended from E. Consequently the condition of equilibrium will be,

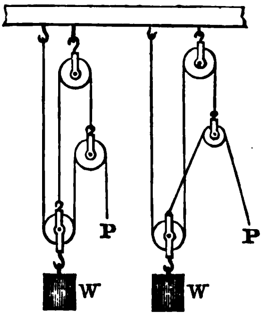
$$\frac{P}{W}=\frac{DL}{EL}.$$

Or if the arms DL, EL be equal, then we must have $P=W$. What I wish you to notice is, that the ratio of P to W is quite independent of the distance of either from the central axis of support; a circumstance which gives this balance great practical advantages, and which, though perhaps at first sight somewhat paradoxical, is not difficult of comprehension when examined carefully.

P. Would it not be a good exercise to discuss all the forces which act upon the different parts of this balance?

T. You cannot have a better; and after what I have said, and the complete manner in which I have explained the preceding weighing machine to you, I think it will offer no great difficulty. First construct a figure representing accurately the action of the forces, remembering that whenever two pieces act upon each other they mutually exert forces which are equal in magnitude, and opposite in direction; and then write down the equations of equilibrium for each constituent member of the machine. And further, I would recommend you, on the first opportunity, to examine a weighing machine of this description, and satisfy yourself that the nature of its construction is such as I have described.

P. The pully appears to supply a very convenient method of increasing our power.



T. It does; and the possible combinations of them are almost infinite, though practically I believe that certain combinations are much more frequently used than others. A combination of pullies is sometimes called a Spanish Barton, or simply a Barton; here are two Bartons upon which you may exercise your skill in determining the relation of P to W.

P. I will endeavour to do so. Can you give me any explanation of the extreme slowness of the operation of raising heavy weights by means of pullies, or similar machinery? or rather, would it not be possible to devise a machine, which should effect its work more rapidly than those in common use?

T. You have here touched upon a very important principle. In considering the conditions of equilibrium of the different systems of pullies, we have of course supposed them, like all the other machines, to be at rest: but *if* you suppose P to descend and W consequently to ascend, you will find in all cases that this result is true, that W will rise through a space which is less than that through which P descends exactly in the same proportion that W is greater than P. Or to put this in a convenient mathematical form, let p be the space through which P descends when W rises through a space w, then you will find that in all cases

$$P \cdot p = W \cdot w.$$

Suppose, for instance, that we have a power of 1 lb. which by means of a Barton is raising a weight of 100 lbs., then in order to raise the weight through 1 foot, the power must descend through 100 feet.

P. Is it difficult to prove the proposition which you have enunciated?

T. Not at all: take the case of the simple moveable pully first. Suppose the weight to rise through 1 inch;

then since the pully from which the weight is suspended hangs by two strings, each of these must be shortened by one inch; therefore *two* inches of string must be given off from the pully, and thus P will descend through *two* inches. Now we know that in the case of the simple moveable pully, $2P = W$, and I have shewn that in this case $p = 2w$, hence $P \cdot p = W \cdot w$. In a similar manner you may extend the proposition to any system of pullies.

P. Consequently, the more powerful a system is, the more slowly it will raise a weight.

T. Certainly; and this principle is commonly expressed by saying that *what is gained in power is lost in time.* This principle applies not only to pullies but to all mechanical combinations whatsoever. I shall not at present enter into it further than to say, that you may notice the application of the principle in such facts as these, that the larger the power-arm of a lever the smaller will be the space described by the end of the weight-arm, when the former is turned through any given angle; when an inclined plane is used for the purpose of enabling us to raise a weight, we obtain greater mechanical advantage by taking a plane of small elevation, though the time required may be longer: thus it is easier to ascend a hill by a long gradual ascent than by a short and steep one; the mechanical advantage of a screw is greater when the angle of its thread is small; and so on. Hence you will perceive that a machine which shall do a certain increased amount of work without a proportional loss of time is a mathematical absurdity. In fact, there never is nor can be, properly speaking, a gain of power by means of a machine; all that the machine can do is to transmit and modify, and if we desire to make a small force effective towards the raising of a very great weight we must do so at the expense of time.

P. This seems to me to throw light upon a remark in p. 39, to the effect, that in the human arm there was a

sacrifice of power, but that the agility of motion gained was much more important.

T. Yes, in this case we may say that what is lost in power is gained in time; a small motion of the point of attachment of the muscle throws the hand through a much larger space, and thus gives a rapidity of motion which could not have been attained had the mere possession of strength been the chief point considered.

P. Are we not now wandering beyond the regions of *Statics?*

T. We *are* and we *are not*. In speaking of the actual motion of machines we *are*, but in speaking of what would be the case *if* the machine were to be moved we *are not*. In fact, the subject which we have now been discussing belongs to what is called the *Principle of Virtual Velocities*, which is a genuine *statical* principle; it is not concerned with the actual motion of a system, but only with what *would* take place if a certain arbitrary motion were impressed upon one part. Thus, in any one of the systems of pullies which we have considered, if we suppose P greater than is required for equilibrium, and we allow it to descend, we shall find that it will descend more and more rapidly, that is, in successive seconds it will pass over longer and longer distances; the determination of the law of this motion belongs to Dynamics: but if, supposing P to be such as to be in equilibrium with W, (in which case it cannot possibly produce any motion of itself,) we conceive P to be drawn through any given space, W must of necessity rise through a certain place on account of the connexion existing between P and W; this motion may be spoken of therefore as merely *geometrical*, and it is this kind of motion which is contemplated in the Principle of Virtual Velocities; in fact, the name *Virtual* excludes the notion of any real dynamical movement. The quantity p is technically called *the virtual velocity of* P, and w *the virtual velocity of* W, and the equation $P.p = W.w$, *expresses the* principle of Virtual Velocities: the principle

is, however, much more general than you might imagine from our present discussion, but the further consideration of it may be most conveniently deferred until you are further advanced in mathematical reading.

P. And in the meanwhile I must be content with the knowledge, that in all machines what is gained in power is lost in time.

T. Yes. I will now take the opportunity of giving you what I promised on a former occasion, namely, Stevinus's proof of the relation of P to W on the inclined plane. It will be perhaps more interesting to you, if I give you the proof precisely as Stevinus gives it himself in his treatise on Statics. In the first book of this treatise he considers the theory of *direct* and *oblique* weights; the fundamental proposition of the theory of the former is the doctrine of the lever, demonstrated after the manner of Archimedes; having solved a variety of problems involving the principle of the lever, he says, "Hitherto have been declared the properties of direct weights; here follow the properties and qualities of oblique, the general foundation of which is contained in the following Theorem."

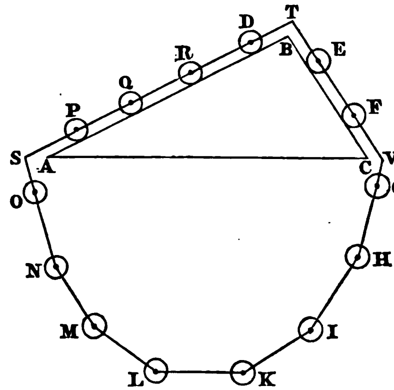
THEOREM. *If a triangle have its plane perpendicular, and its base parallel, to the horizon; and upon the two sides be placed two equal spherical weights; as the right hand side of the triangle is to the left, so will be the power of the left hand weight to that of the right.*

Let ABC be a triangle having its plane perpendicular to the horizon, and its base AC parallel to the same: and upon the side AB (which is double of BC), let there be placed a globe D, and upon BC another, E, equal to the former in weight and magnitude.

It is required to prove, that as $AB : BC$, *i. e.* as $2 : 1$, so is the power of the weight E to that of D.

7—2

Let there be arranged round the triangle 14 globes, equal in weight and magnitude, and equidistant, strung upon a line passing through their centres, in such a manner that they may be able to turn about the said centres, and that there may be two globes upon the side *BC*, and four upon the side *AB*; then as one line is to the other, so is the number of globes to the number of globes.



Also at *S*, *T*, *V* let there be three fixed points, upon which the line or thread may be able to run, and let the two portions above the triangle be parallel to the sides *AB*, *BC*; so that the whole may be able to turn freely and without catching upon the said sides *AB*, *BC*.

DEMONSTRATION.

If the power of the weights *D*, *R*, *Q*, *P*, be not equal to that of the two globes *E*, *F*, the one side will be more powerful than the other; let (if possible) the four *D*, *R*, *Q*, *P* be more powerful than the two *E*, *F*; but the four *O*, *N*, *M*, *L* are equal in effect to the four *G*, *H*, *I*, *K*; wherefore the eight globes *D*, *R*, *Q*, *P*, *O*, *N*, *M*, *L*, will be more powerful than the six, *E*, *F*, *G*, *H*, *I*, *K*; and since the stronger power must prevail over the weaker, the eight globes will descend, and the six will rise. Let this be so, and let *D* arrive at the place which is at present occupied by *O*, and so of the rest; thus *E*, *F*, *G*, *H* will assume the places occupied by *P*, *Q*, *R*, *D*, and *I*, *K* those occupied by *E*, *F*; notwithstanding the change therefore, the globes will have the same disposition as before, and for the same reason the eight globes will descend in virtue of their *superior gravity*, and in descending will cause eight other

to take their places, and so the motion will be perpetual; which is absurd. And the like demonstration will hold for the other side: the set of globes *D, R, Q, P, O, N, M, L* will therefore be in equilibrium with *E, F, G, H, I, K.* Now take away from the two sides those weights which are equal and similarly situated, as are the four globes *O, N, M, L* on the one side, and the four *G, H, I, K* on the other; the remaining four *D, R, Q, P* will be, and will continue to be, in equilibrium with the two *E, F*: wherefore *E* will have a power twice as great as that of *D*; as therefore the side *AB* to the side *BC*, or as 2 : 1, so is the power of *E* to the power of *D*. Therefore, *If a triangle*, &c. Q. E. D.

Such is Stevinus's demonstration; from it he deduces the doctrine of the inclined plane in its most general form, that is in fact the laws of oblique forces. I shall not, however, trouble you with any further extracts from his work; but the fundamental proposition, which I have just now given you, is interesting from its ingenuity, and because it was the first independent proof of the laws of oblique forces.

P. When Stevinus speaks of the *power* of a weight, I suppose he means what we should call the *resolved part* of the weight.

T. Certainly. The weight of any one of Stevinus's globes may be resolved into two parts, one perpendicular and the other parallel to the side of the triangle upon which it rests; the former produces pressure upon the side, the latter tends to produce motion in the string of globes; the part of the weight of each globe which is thus effective in producing motion, or tending to produce it, is what he calls the *power* of the globe.

P. It is taken for granted that the four globes *G, H, I, K* will be similarly situated to *O, N, M, L*; is it not?

T. Yes, this is assumed; and I think it is a fair assumption; because it seems impossible to assign any

reason for the one set of globes arranging themselves in a manner different from the other; what the actual arrangement will be we do not care to know. You will observe, that as the globes below the triangle are upon a thread which passes over the two fixed points *S* and *V*, the arrangement of those globes must be altogether independent of the unsymmetrical arrangement of the six globes above; in fact, so far as the arrangement of the eight lower globes is concerned, you may, if you please, suppose the thread to be fastened at the two points *S* and *V*, and then the question will be reduced to that of eight equal and equidistant globes strung upon a thread, the extremities of which are fastened to two points in the same horizontal line; there can be no doubt that these globes will arrange themselves symmetrically, four upon each side of the vertical line passing through the middle point between the fixed points of suspension.

P. I feel quite disposed to grant this. There is another assumption in the proof, viz., that the perpetual revolution of the globes in the manner described is impossible.

T. There is, and the whole proof turns upon it. There can be no doubt, I think, that the notion of the strings of globes keeping each other in perpetual motion does at once strike us as absurd; if we inquire why this is the case, perhaps we may be led to the conclusion from the observation, that all systems of bodies upon the earth's surface tend to find for themselves a position of rest; if we arrange a number of bodies in any way, and connect them in any way, and then leave them to themselves, motion may or may not ensue, but if it does the bodies after falling soon arrange themselves in a position of rest and so remain; in fact, gravity always tends one way, that is, it tends to make bodies descend and not ascend, so that when the mass acted upon has on the whole descended as much as the connexion of its parts will allow, there is no force tending to raise it again.

P. But would not this reasoning shew that if a ball were placed in a bowl, not at the bottom, it would run down to the bottom and there rest?

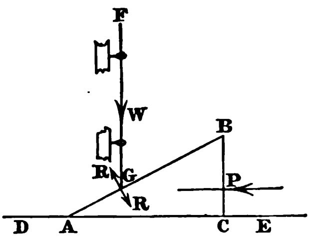
T. No; it is true that it would run down to the bottom and that then there would be no force tending to raise it again; it will however run past this position in consequence of the motion which it already has, and run up the other side of the bowl, then descend; and so on perpetually, as you will understand more clearly when you have studied Dynamics. But the difference between this case and that of Stevinus is this, that here the effect of gravity is actually to depress the matter of the ball, and this depression causes the ball to move even at the bottom of the bowl where gravity has no power to move it; but in the case of Stevinus's balls there is no such depression of the matter of the system; if the balls revolve the matter is still arranged exactly as it was before: now this is the kind of effect which gravity cannot produce.

P. I believe I understand this, and the more so because we discussed a similar question on a former occasion (p. 86.)

T. Let us then now dismiss Stevinus; and the subject of the inclined plane reminds me that I should notice, that in the chapter which you have just now been reading, I have not introduced the *Wedge* which is usually included amongst the Mechanical powers. The wedge is in fact only a moveable inclined plane; in order that the wedge may be useful in the construction of machines, it is necessary that it should be constrained to move in some assigned manner, and as soon as the manner in which it is possible for it to move is assigned the conditions under which it can be in equilibrium can be investigated without difficulty.

For instance; let ABC be a wedge moveable upon the smooth horizontal plane DE, and let C be a right angle, and $BAC = \alpha$. Also suppose that a heavy vertical

beam FG passing through two rings, rests upon the surface AB of the wedge at G; then the pressure upon this surface would cause the wedge to slide, if not prevented, along the plane DE; let it be prevented from doing so by a horizontal force P, then it is required to find the ratio of P to the weight W of the beam.



There will be a mutual action between the beam and the surface of the wedge, which will be perpendicular to that surface if we suppose it to be perfectly smooth; call this action R. Then for the equilibrium of the beam, we have, resolving the forces vertically,

$$R \cos a = W, \ldots\ldots (1).$$

And for the equilibrium of the wedge, resolving horizontally,

$$R \sin a = P; \ldots\ldots (2).$$

$$\therefore \frac{P}{W} = \tan a,$$

which is the relation required.

P. What becomes of the horizontal part of R in the equilibrium of the beam?

T. I have omitted it, because it will be in equilibrium with the horizontal pressures of the rings upon the beam; in fact besides the equation (1) there are two other equations of equilibrium for the beam, the horizontal equation, and the equation of moments; these will give us the pressure on the two rings, which will be both horizontal, since on account of the supposed smoothness of the beam they must be perpendicular to its length. Let me now ask *you*, what becomes of the vertical part of R in the equilibrium of the wedge?

P. I suppose it will be counteracted by some other force which you have omitted.

T. Yes, by the pressure of the horizontal plane *DE*. The other two equations for the wedge (besides [2]) will determine the magnitude of this force and the point at which it acts.

P. It appears from the result, that the mechanical advantage will be greater when the angle of the wedge is small than when it is large; this might have been anticipated.

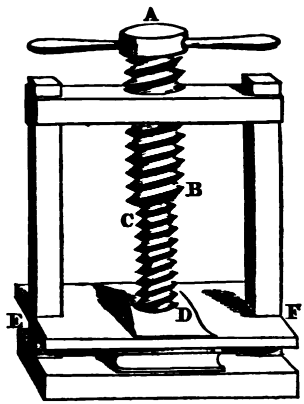
T. It might have been; and this will be the case in all applications of the wedge, though it will not always be true that the mechanical advantage will depend upon the tangent of the angle only, as in this case. It will not be so for instance, in the common and very rough application of the wedge to split a block of wood; in this case the force corresponding to *W* is the resistance of the block to splitting, which acts in a direction which it is very difficult or perhaps impossible to ascertain: on this account I shall not trouble you with the investigation, which is in reality of no practical importance, as you will easily believe when you consider the very coarse and unscientific character of the operation to which it refers. The wedge is, however, under certain forms of great utility; especially I may remind you that a great number of cutting instruments are wedges, as for example the chisel: in constructing such instruments two things have to be borne in mind; the mechanical power of the tool, as we have seen, is increased by diminishing its angle, but then the strength of the tool is diminished at the same time. Practically a balance has to be struck between the two, and the result will depend upon the degree of hardness of the substance upon which the tool is to be used; for wood the angle may be about 30°, for iron about 50° or 60°, for brass still larger.

P. I perceive that the wedge affords another illustration of the principle that what is gained in power is lost in time.

T. It does, and I will now conclude this long conversation by describing to you a machine which furnishes a

still better illustration. The machine of which I speak is *Hunter's Screw;* you will remember that the power of a screw was shewn to depend upon the smallness of the interval between the threads, so that in order to have a very powerful screw we must have a very fine thread; the difficulty is, that if the thread be made too fine, there is danger of its not being able to resist the pressure upon it, and being thus torn from the cylinder. This difficulty is avoided by means of the compound screw, which I am now going to describe.

AB is a screw which passing through a block may be worked by a lever in the usual manner; but instead of pressing immediately upon the board *EF*, the screw *AB* acts upon another screw *CD* which enters the former by means of an interior thread cut to fit it. The screw *AB* is slightly coarser than the screw *CD*, so that when *AB* descends through a given space, *CD* ascends through a space not quite so great, and the board *EF* is pressed through a space equal to the difference of the spaces respectively passed through by the two screws.



Now upon the principle that what is lost in time is gained in power, we may conclude that as in the common screw the mechanical advantage depends upon the distance between the threads, so in this compound screw the mechanical advantage will depend upon the difference of these distances for the two component screws.

Suppose, for instance, that D is the distance between the threads for AB, and d for CD; then one turn of the lever will make AB descend through a space D, and CD ascend through a space d; so that on the whole EF will advance through a space $D - d$, and the reciprocal of this quantity $D - d$ will measure the power of the screw.

P. And D and d can be made as nearly equal as we please.

T. Yes: that is the advantage of the contrivance; and yet D and d may themselves be large, or the component screws may be as coarse as we please. Suppose, for instance, that AB has three turns in an inch, and CD four; so that $D = \frac{1}{3}$, $d = \frac{1}{4}$, then $D - d = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$: and therefore the power of the compound screw is to the power of the upper of the two component screws as 12 : 3, or it is four times as great.

I may mention that this same principle may be applied by cutting two screws upon the same cylinder, and passing these through two nuts or blocks, which are capable of approaching each other, but are not allowed to revolve. Then it is evident that a turn of the cylinder will cause the nuts to approach by a space equal to the difference between the breadths of the threads of the two screws, and thus the same effect will be produced as in the construction which I have before described.

EXAMINATION UPON CHAPTER VI.

1. The arms of a false balance are respectively 1 foot, and 1.05 in length; what will be the apparent value of 100 lbs. of tea, weighed out with such a balance, (the tea being suspended from the longer arm), if the shop-keeper undertake to sell his tea at 4*s*. per pound?

2. If when a balance is suspended the beam be not horizontal, prove that if the want of horizontality arise from an inequality in the weight of the scale-pans, the balance may be corrected by putting a weight into the lighter of the two, but that if it arise from a difference of length of the arms the balance cannot be so corrected.

3. Shew how to graduate the common steelyard.

4. Shew how to graduate the Danish steelyard, in which the fulcrum is moveable.

5. Find the relation of P to W in the single moveable pully.

6. In the first system of pullies.

7. In the second.

8. In the third.

9. Find the relation of P to W in the inclined plane.

10. In the screw.

11. What force is necessary to support a weight of 50 lbs. upon a plane inclined at an angle of 30° to the horizon, the force acting horizontally?

12. What force is necessary in the preceding problem, if the force act vertically?

13. When a given weight is sustained upon a given inclined plane by a force in a given direction, find the pressure upon the plane.

14. Given the weight, and the magnitude and direction of the sustaining force, find the inclination of the plane.

15. On an inclined plane the pressure, force, and weight, are as the numbers 4, 5, 7; find the inclination of the plane to the horizon, and the direction of the force.

16. What weight is that which it would require the same exertion to lift as to sustain a weight of 4 lbs. upon a plane inclined at an angle of 30^{0} to the horizon?

17. A weight W is sustained upon an inclined plane by a force P, acting by means of a wheel and axle, placed at the top, in such manner that the string attached to the weight is parallel to the plane. Given R and r the radii of the wheel and axle, find the inclination of the plane.

18. Two weights sustain each other upon two opposite inclined planes, by means of a string which is parallel to the planes; compare the pressures on the planes.

19. What force must be exerted to sustain a ton weight on a screw, the thread of which makes 100 turns in the course of 12 inches, and which is acted upon by an arm 4 feet long?

20. Find the inclination to the horizon of the thread of a screw, which with a force of 5 lbs. acting at an arm of 2 feet, can support a weight of 300 lbs. on a cylinder of 2 inches radius.

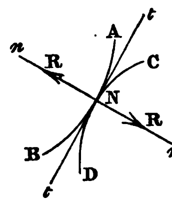
If the length of the cylinder be 4 feet, find the entire length of the *thread of the screw.*

CHAPTER VII.

ON FRICTION.

---

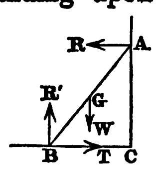
1. In the preceding chapter we had occasion to speak of the effect of a smooth plane upon a body in contact with it, and we concluded that the effect would be to produce a force upon the body in the direction perpendicular to the plane. Upon the same principle we can conclude the nature of the force which exists between any two smooth surfaces in contact. Let *AB, CD* be two smooth surfaces in contact at *N*; then they will exert upon each other a pressure, which (as we do not know its magnitude) we will call *R*. This pressure will be mutual; that is, if *AB* presses against *CD* with a force *R, CD* must of necessity press in the exactly opposite direction with an equal force: this is manifest from the nature of the case. The only question is, in what direction will the forces *R* act? Now since the surfaces are in contact at *N*, they must have a common tangent at that point; let it be *tNt*; draw *nNn* perpendicular to this tangent, then we call this line a *normal* to the surfaces, or the common normal. Since the surfaces are *smooth* they cannot exert any action upon each other in the direction *tNt*, therefore the whole action must be in the direction *nNn*, or *in the direction of the common normal.*



2. Although in practice no surface is perfectly smooth, yet the simplicity gained for mathematical investigations by the supposition of perfect smoothness is so great, that problems are usually solved with this imaginary condition.

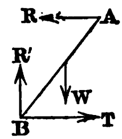
In the chapter upon machines, for instance, we have adopted this method. Consequently, it is necessary in solving problems to be able to determine at once the manner in which smooth surfaces act upon each other; and even if we take the actual case of bodies having some degree of roughness, it will be found that the nature of the mutual action in that case will be best understood by first becoming familiar with the simpler case of smooth surfaces.

3. One of the most common cases of mutual action is that of a particle, or a surface of indefinitely small magnitude, the end of a rod for instance, upon another surface, as that of a plane, a sphere, or the like. In this case it is to be observed, that the surface of finite magnitude determines the direction of the mutual pressure, because any line whatever may be considered as *normal* to the surface of a point. The meaning of this will be seen better from examples.

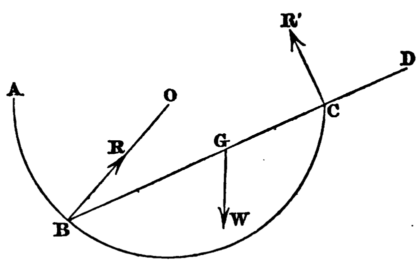


Let AB be a pole, a rod, or ladder, standing upon the smooth horizontal plane BC and resting against the smooth vertical wall AC; we must suppose the foot of the ladder restrained by a string BC, otherwise it will evidently slide down. Now first what will be the action of the wall upon the ladder at A? It will be a pressure of unknown magnitude, R, normal, that is perpendicular, to the wall. Again, what will be the action of the smooth horizontal plane at B? It will be a pressure of unknown magnitude, R', normal, that is perpendicular, to the horizontal plane. Let us complete the consideration of the forces acting upon AB. The string BC will produce a force, which we call the *tension* of the string, in the direction BC: this we will denote by T. And lastly, we have the weight of AB, which we may regard as one single force W, acting vertically at the centre of gravity G. Hence *the problem is that* of a rod AB, under the action of the

forces represented in the accompanying figure; and we may divest ourselves of all notion of wall, ground, &c., and confine our attention to the system of forces as there represented.



Let us take another example: *BD* is a smooth heavy rod or beam, resting (as represented in the figure) in a fixed smooth hemispherical bowl *ABC*, the centre of which is *O*. There will be an action of the bowl upon the rod at *B* and *C*, which we will denote by *R* and *R'* respectively: what will be their directions?



At *B*, only the extremity of the rod, which we regard as a point, rests upon the surface of the bowl; consequently the pressure *R* will be perpendicular to the tangent at *B*, that is, its direction will pass through the centre of the sphere *O*. But at *C* the surface of the beam is in contact with the edge of the bowl, which we regard as indefinitely thin; consequently the pressure *R'* will be perpendicular to *BD* the direction of the beam.

In this case, as in the preceding, having once determined the directions of the various forces, and represented them, as in the figure, we dismiss the consideration of the bowl, and confine our attention to the equilibrium of the beam *BD*, acted upon by the three forces *R*, *R'*, *W*.

The instances which we have now considered will be sufficient for our present purpose; the subject will be further illustrated in the next chapter, in which we shall be employed in solving problems, and shall shew by what conditions the unknown pressures, of which we have been speaking, must be determined.

4. We now pass to the case of *rough* surfaces. When a force tends to draw one body over the surface of another,

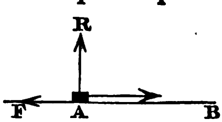
there is, as we know by constant experience, a resistance to motion; and this resistance we call the force of *friction*. Its intensity manifestly depends partly upon the nature of the surfaces in contact; thus it is more easy to drag a load over a wooden floor than over a turnpike road, still more easy over a polished marble floor, and the resistance of this last may again be diminished by throwing upon it a small quantity of oil. Now, inasmuch as in all practical problems the force of friction enters in a very important manner, it has been the business of scientific men to determine as far as possible by experiment what are the laws of its action. The following are the conclusions to which they have been led.

5. Suppose two bodies to be in contact, and one of them to be on the point of sliding over the surface of the other in consequence of some force acting upon it, and suppose it to be restrained from motion by the force of friction, then

(1) The force of friction is proportional to the mutual pressure between the two bodies; and

(2) The force is independent of the extent of the surface in contact.

Suppose, for instance, that a body A rests upon a plane surface, and that the pressure exerted upon the surface by its own weight or otherwise is R; and suppose it to be on the point of moving in the direction AB under the action of any force, and to be restrained from moving by the opposite force of friction F. Then F is proportional to R, or $F \propto R$; that is, if we double R, we double F; a weight of 2 lbs. will offer twice as much resistance to motion over a rough horizontal plane as a weight of 1 lb.; a weight of 3 lbs., three times as much, and so on.



This relation is generally expressed by the equation

$$F = \mu R,$$

where μ is a quantity called the *coefficient of friction*, and depends upon the nature of the surfaces in contact; thus, for wood it will have one value, for slate another, ivory another, and so on. It will also depend upon the degree of polish which the surfaces may have, the absence or presence of any oily matter between the bodies, and so on.

The second law teaches us, that provided the pressure is the same the friction does not depend upon the extent of surface in contact; that is, a weight of 1 lb. will not exert more friction if it rests upon a surface of two square inches than if it rests on a surface of one. This law however is not true in extreme cases, as when the surface in contact is reduced very nearly to a point.

On the whole, the formula $F = \mu R$, will express all the laws of friction, if we remember that μ is a quantity independent of R, independent of the extent of surface in contact, and dependent only upon the nature of the surfaces.

6. It will be remembered, that the friction of which we here speak is that friction which exists when the two surfaces with which we are concerned are *on the point* of sliding one upon the other; this is called *Statical Friction.* When motion actually takes place the amount of friction is in general different; this is called *Dynamical Friction*, and will not occupy our attention at present; we may however mention in passing, that the same two laws hold for Dynamical as for Statical friction, but that there is in addition this law, that the friction is independent of the velocity with which the surfaces move.

The following are a few results deduced by experiment:

Wood upon wood, $\mu = .50$.
Wood upon metal, $\mu = .60$.
Metal upon metal, $\mu = .18$.

These values may be much reduced by introducing

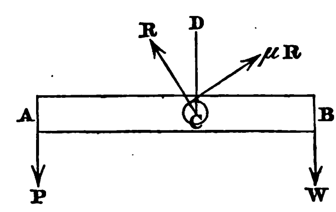
between the surfaces any oily substance; for instance, in the last case, if olive oil be introduced between the metals, $\mu = .12$.

7. Forasmuch as friction depends entirely upon the mutual normal pressure, the consideration of friction does not introduce any new unknown force into a problem. And in general when a problem can be solved upon the supposition of the surfaces involved in it being smooth, it can also be solved with a little increase of trouble, but with no real increase of difficulty, upon the supposition of the surfaces being rough, at least, it can be solved *for the state bordering on motion.* In the machines, for instance, of the preceding chapter, friction is practically a very important force, and one which may by no means be omitted; and it will be found, after what has been said, that there is no difficulty in introducing the consideration of it, provided only that we suppose P to be on the point of descending, or on the point of ascending; these will form two limiting cases between which all other possible cases will be included.

8. We will now give a few examples of the solution of problems with friction.

Ex. 1. The lever with friction.

Let AB be a lever working upon a cylindrical axis C; then if we suppose friction to exist between the surface of the axis and the aperture in which it works, and suppose that the lever is horizontal, but the extremity B on the point of descending, the pressure upon the axis will not be in the vertical direction CD, but inclined to it at some angle θ, (suppose). Call this pressure R, then we shall have a force of friction μR, (as in the figure), tangential to the cylindrical axis, that is, perpendicular to the direction of R.



Let $AC = a$, $BC = b$, and let r be the radius of the cylindrical axis; then, by resolving horizontally and vertically and taking moments about C, we shall have the following three equations;

$$R \sin\theta - \mu R \cos\theta = 0 \text{ ....} (1),$$
$$R \cos\theta + \mu R \sin\theta = P + W \text{ ....} (2),$$
$$Pa = Wb + \mu Rr \text{ ....} (3).$$

Equation (1) gives us,

$$\tan\theta = \mu \text{ ....} (4),$$

and from (2) and (3) we have,

$$\mu r (P + W) = (\cos\theta + \mu \sin\theta)\, \mu Rr,$$
$$= (\cos\theta + \mu \sin\theta)(Pa - Wb);$$
$$\therefore P\{a(\cos\theta + \mu \sin\theta) - \mu r\} = W\{b(\cos\theta + \mu \sin\theta) + \mu r\},$$
$$\text{or } \frac{P}{W} = \frac{b(\cos\theta + \mu \sin\theta) + \mu r}{a(\cos\theta + \mu \sin\theta) - \mu r}.$$

If we put for μ its value from equation (4), we have,

$$\frac{P}{W} = \frac{b(\cos\theta + \sin\theta \tan\theta) + r \tan\theta}{a(\cos\theta + \sin\theta \tan\theta) - r \tan\theta}$$
$$= \frac{b + r \sin\theta}{a - r \sin\theta} \text{ ....} (5)$$

In applying this formula we may find θ from equation (4) by means of a trigonometrical table, and then put the value of $\sin\theta$ in (5). And it may be remarked, that the equations of problems, in which friction is involved, generally assume their simplest form when we represent the coefficient of friction by the tangent of a certain angle. On the other hand, we may, if we please, express the ratio of P to W in terms of μ, and we have then

$$\frac{P}{W} = \frac{b + r\dfrac{\mu}{\sqrt{1+\mu^2}}}{a - r\dfrac{\mu}{\sqrt{1+\mu^2}}}.$$

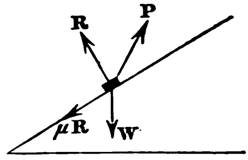
We have supposed that P is on the point of descending; if on the contrary, W be on the point of descending, we should find in like manner that

$$\frac{P}{W} = \frac{b - r\dfrac{\mu}{\sqrt{1+\mu^2}}}{a + r\dfrac{\mu}{\sqrt{1+\mu^2}}},$$

a result which may be obtained from the other *by changing the algebraical sign of* μ.

Ex. 2. The inclined plane with friction.

We will take the most general case, in which P acts in a direction making any angle ϵ with the inclined plane; and we will suppose the weight W to be on the point of *ascending;* then the forces will be such as represented in the figure. If α be the angle of the plane, we shall have by resolving parallel and perpendicular to the plane,



$$P\cos\epsilon - W\sin\alpha - \mu R = 0 \text{ .... } (1)$$

$$P\sin\epsilon - W\cos\alpha + \mu R = 0 \text{ .... } (2)$$

Multiplying equation (2) by μ and adding it to (1) there results

$$P(\cos\epsilon + \mu\sin\epsilon) = W(\sin\alpha + \mu\cos\alpha),$$

$$\therefore \quad \frac{P}{W} = \frac{\sin\alpha + \mu\cos\alpha}{\cos\epsilon + \mu\sin\epsilon} \text{ .... } (3)$$

If W be on the point of descending, the force of friction will be in the opposite direction, and the result will be

$$\frac{P}{W} = \frac{\sin\alpha - \mu\cos\alpha}{\cos\epsilon - \mu\sin\epsilon} \text{ .... } (4)$$

If, according to the remark made in the last example, we put $\tan\beta$ instead of μ, where β is a *subsidiary angle* determined by the equation

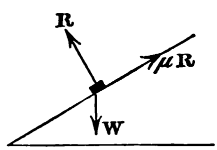
$$\tan\beta = \mu,$$

then (3) and (4) become respectively,

$$\frac{P}{W} = \frac{\sin(\alpha+\beta)}{\cos(\epsilon-\beta)}, \text{ and } \frac{P}{W} = \frac{\sin(\alpha-\beta)}{\cos(\epsilon+\beta)}.$$

By making $\beta = 0$ we reduce these expressions to those already investigated in page 131 for the case of the smooth inclined plane.

Ex. 3. If in the last result we make $P = 0$, we have $\alpha = \beta$, or the subsidiary angle β is the inclination of the plane for which a weight will just not slide. Let us, as another example, determine directly the angle at which a plane may be inclined without allowing a weight placed upon it to descend.



The forces will be as in the figure; resolving horizontally and vertically, we have

$$R\sin\alpha - \mu R\cos\alpha = 0 \text{ .... } (1)$$

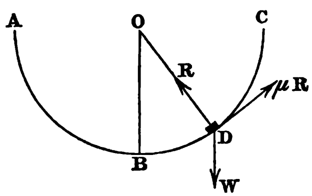
$$R\cos\alpha + \mu R\sin\alpha = W \text{ .... } (2);$$

from (1) there results,

$$\tan \alpha = \mu,$$

which gives us the angle required. Equation (2) gives us the value of R.

Ex. 4. If the interior surface of a hemispherical bowl be perfectly smooth, it is manifest that a particle cannot rest except at the lowest point of the bowl; but if the surface be rough, the particle may rest at a distance from the lowest point: let us determine the limits within which this is possible.



Let ABC be the bowl, O its centre, OB vertical, D the furthest point from B at which a weight W will rest, $BOD = \theta$. The forces will be as in the figure. Then resolving horizontally, we shall have

$$R \sin \theta - \mu R \cos \theta = 0,$$

$$\text{or} \quad \tan \theta = \mu,$$

which equation determines the position of D.

9. These examples will be sufficient to illustrate in general the method of treating problems, when the surfaces involved in them are not perfectly smooth; and they will shew that no new unknown forces are introduced into problems by the consideration of friction, provided we take only the extreme case in which the surfaces in contact are on the point of sliding one upon another. In the next chapter we shall meet with other problems which will further illustrate the subject.

CONVERSATION UPON CHAPTER VII.

P. It appears from this chapter that bodies with smooth surfaces, such as we have hitherto considered, are merely mathematical fictions.

T. Yes: a *perfectly* smooth body does not exist, and the greater number of surfaces are very far from smooth.

Nevertheless you will perceive that it is desirable to treat of smooth surfaces in the first instance, because their theory is so much more simple than that of rough surfaces, and we can very easily pass from thence to the other.

P. The case which has been solved, however, appears to be only the particular case in which the surfaces are on the point of sliding.

T. It is only that limiting case; but from the result we can generally deduce any practical conclusions which we may desire. For instance, we have determined the value of P for a given value of W upon the inclined plane, when W is on the point of ascending, and when it is on the point of descending, then we know that the value of P must always lie between these two extreme values, and this will generally be all that will be required. Again, we know that a weight will just rest upon an inclined plane if the angle of inclination be given by the equation $\tan\theta = \mu$; then if the inclination be less, the weight will rest *a fortiori*. Again, if a particle rest upon a horizontal plane, any force acting upon it must be perpendicular to the plane, but if the plane be rough this will not be the case; suppose the particle W to be on the point of moving, and θ to be the angle which the direction of the supposed force P makes with the vertical, and let R be the normal pressure, μR the force of friction; then resolving vertically and horizontally, we have

$$P\cos\theta + R = W,$$

$$P\sin\theta = \mu R,$$

$$\therefore\ P(\mu\cos\theta + \sin\theta) = \mu W:$$

this equation determines θ, and we conclude that the particle will be at rest, provided the direction of P lie within a cone of which the axis is vertical and the semivertical angle the angle θ determined as above. And so in other instances.

P. I should suppose from what we have been reading, *that friction is* a very important force in nature.

T. Most important; the climbing a hill, or even walking upon level ground, would be practically impossible without it; for in this latter case you will easily see that the pressure of the foot upon the ground must, unless for this action of friction, be accurately normal to the horizon, a condition which in practice could not be satisfied; to take another instance, when a nail is driven into a piece of wood, it would not remain in its place if it were not for the force of friction; in these and in hundreds of everyday instances, friction is a most useful force. Then, on the other hand, in the construction of machines friction is often far from useful, one great difficulty with which mechanicians have to contend being in some cases that of doing away with the effects of friction; the construction of clocks and watches is a good example.

And you will be familiar with common instances in which devices are adopted for the purpose of diminishing resistance due to this cause: thus you will see a large block of stone transported by placing rollers underneath it, the intention being to diminish the enormous amount of friction which would exist between the heavy block and the ground. The wheels of carriages are the same device in a more delicate form; in this case the friction is reduced to that of the polished surface of the axle, which by a constant supply of oil or grease is made very small. And there is the very ingenious device of *friction wheels;* to understand this invention, let *AB*, *CD* be two wheels, turning about horizontal axes *P*, *Q*; and let there be upon the same axle *Q* a third wheel (not seen in the figure) behind *CD* and exactly similar to it;

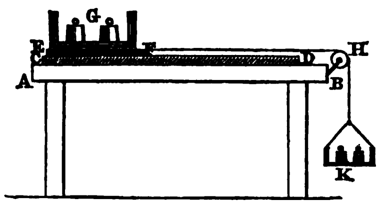
lastly, let *EF* be a wheel which we desire to cause to turn with as little friction as possible; then its axle *O*, instead of running in a socket, is allowed to rest upon the surfaces of the three friction wheels just described. The consequence of this arrangement is, that any tendency to friction between the axle *O* and the surface of the friction wheels, instead of being resisted by a fixed surface, causes the friction wheels

to revolve, and thus the amount of friction is diminished to a very remarkable extent.

P. How are the laws of friction proved?

T. I will describe a simple apparatus, by means of which experiments can be made.

AB is a firm horizontal table, upon which can be fastened a flat piece *CD* of one of the substances concerning which we desire to make experiments; *EF* is the other substance, which lies upon the surface of *CD*, and bears upon it the case



G which may be weighted to any extent. A horizontal string *FH*, attached to *EF*, passes over a pully at *H* and carries a scale *K* which also may be weighted as we please. In order to make experiments upon the friction between two given substances *CD* and *EF*, we have only to put a certain weight in *G*, and load the scale *K* until *EF* just begins to move; then it will be seen that the weight of the case *G* and its contents, added to the weight of *EF*, measures the pressure between the substances, and the weight of the scale *K* and its contents measures the friction for the state bordering on motion. It will be easy to vary the amount of the weight in *G*, the nature of the surfaces in contact, and the extent of the surfaces; the result of experiments so made is to establish the laws of friction which have been given in this chapter.

It should be remarked, that in making experiments with soft bodies, such as wood for instance, the amount of friction will depend to a certain extent upon the time during which the surfaces have been in contact, being less at first than it is afterwards. This is quite what we should expect; the increase of friction only takes place during a limited time: thus with wood upon wood, for instance, the friction attains its greatest value after about two or *three minutes*; for wood upon metals a longer time is

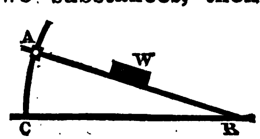
required for the friction to attain its permanent value, sometimes even as much as several days.

P. The method of making the experiments seems very simple, and the results of course cannot be questioned, but I confess that one of the laws appears to me slightly paradoxical; I mean that which states that the friction between two surfaces is independent of the extent of surface in contact.

T. The result is not so strange as it may appear at first sight. For suppose a body of a certain weight rests upon another with a certain extent of surface in contact, and suppose that without altering the weight we diminish the surface in contact by one half, then if we regard the pressure as uniformly distributed over all the points of the surface in contact, we shall have diminished the number of points by one half, but we shall have *doubled the pressure* upon each point, since upon the whole the same pressure has to be sustained as before. And since the friction varies directly as the pressure, the *friction* at each point will be *doubled.* Hence on the whole as we diminish the extent of surface in contact, we increase the intensity of the friction, and therefore the friction will be *coet. par.* independent of the extent of surface.

P. I did not view it in that light before. Is it by such an apparatus as you have described that the value of μ would be actually determined for various substances?

T. It might be so determined, but perhaps the results may be obtained more simply thus. You will remember that it has been shewn that if a be the greatest angle of inclination of a plane of given substance upon which a portion of another given substance can rest, and if μ be the coefficient of friction for the two substances, then $\tan a = \mu$. Hence to determine μ it will be sufficient to determine a; suppose now that AB is a bar moving about a hinge B in a horizontal bar



8

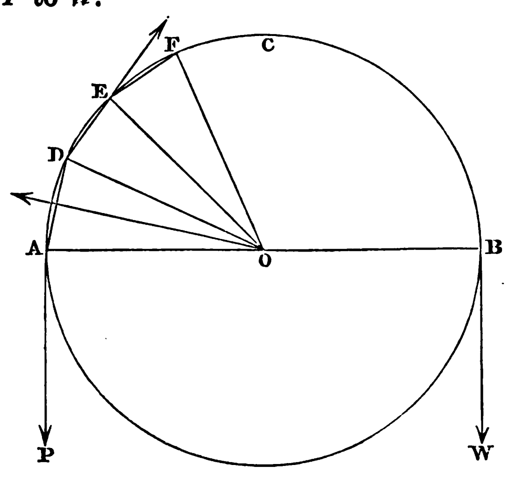
BC, and that CA is a graduated circular rim; suppose also that upon AB we can screw a plate of one of the substances, and upon this plate let the other substance W be placed; then let the extremity A be raised until W is on the point of descending; ABC will be then the angle a, and will thus give us the value of μ.

P. There is one other question which I should wish to ask respecting friction: it has always seemed surprising to me that the force of friction should be so much increased by twisting a rope several times round a post; I am struck with this upon almost every occasion that I am on or near the sea, for sailors are well aware of the fact, and take great advantage from it.

T. The result is certainly striking; it admits however of mathematical explanation, and is treated of in all books which deal with Statics by help of the Differential Calculus. I shall however be able if you give me your attention, to make the cause apparent to you, and even to give you the exact mathematical expression for the increase of mechanical advantage.

Let us consider this problem; two weights, P and W, are connected by a string which passes over a given rough horizontal cylinder: P is on the point of descending, required the ratio of P to W.

Let ACB be a section of the cylinder; O its centre. Then the string connecting P and W will be in contact with the cylinder throughout the semicircumference ACB.



Now the difficulty of the problem consists in

this, that the direction of the string upon the cylinder is constantly changing its direction from point to point: in order to overcome this difficulty let us cut up the arc ACB into n small equal parts, $AD, DE, EF,\ldots\ldots$; join $AD, DE, EF,\ldots\ldots$ and, instead of considering the circular arc ACB, let us consider the polygon $ADEF\ldots\ldots$ and let us suppose the string to rest upon a prismatic post of many sides instead of a cylindrical post.

Let us consider how the piece of string AD is held in equilibrium; there will be the force P at A, and a force at D in the direction of DE, which will be composed partly of the tension of the string, which we will call P_1, and partly of the friction, which we will call F. There will be a third force, or rather a system of forces perpendicular to AD arising from the pressure of the surface of the post upon the string, but these forces we may consider to be equivalent to one single force acting at the middle point of AD perpendicular to it and proportional to its length; this force we will call therefore $p.AD$. Lastly, it will be observed that the angle AOD is by construction equal to the n^{th} part of two right angles, or $\frac{\pi}{n}$.

Now for the equilibrium of AD we have, resolving parallel and perpendicular to it,

$$P = P_1 + F \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$p\,.\,AD = P\cos DAO + (P_1 + F)\cos ADO \ldots\ldots (2),$$

and by the law of friction

$$F = \mu\,.\,P.AD \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

$$\text{Now } \cos DAO = \cos ADO = \sin\frac{AOD}{2} = \sin\frac{\pi}{2n},$$

$$\text{also } AD = AO \text{ chd. } AOD = 2AO\sin\frac{\pi}{2n};$$

$\therefore$ equations (1), and (2), give us,

$$2pAO\sin\frac{\pi}{2n} = 2P\sin\frac{\pi}{2n};$$

$$\therefore p = \frac{P}{AO}.$$

$$\therefore \text{ from (3), } F = \mu P . \frac{AD}{AO} = 2\mu P \sin\frac{\pi}{2n};$$

and therefore from (1),

$$P_1 = P - 2\mu P \sin\frac{\pi}{2n} = P\left\{1 - 2\mu\sin\frac{\pi}{2n}\right\}.$$

In like manner if P_2 be the tension of the string at E, we have

$$P_2 = P_1\left\{1 - 2\mu\sin\frac{\pi}{2n}\right\}$$

$$= P\left\{1 - 2\mu\sin\frac{\pi}{2n}\right\}^2.$$

And so on; but we have divided the circumference ACB into n parts, consequently, according to our notation,

$$W = P_n = P\left\{1 - 2\mu\sin\frac{\pi}{2n}\right\}^n.$$

This formula will explain to you the advantage of wrapping a cord upon the surface of a many-sided prismatic post, from which the case of a cylinder is not a very difficult step. For, suppose we have a prismatic post of $2n$ sides; and let P and Q be the two forces which are in equilibrium at the extremities of a rope wrapped upon it, P being on the point of overcoming Q; then it will appear, that if we make for shortness' sake $\left(1 - 2\mu\sin\frac{\pi}{2n}\right)^{2n} = r$,

for one complete turn upon the post, $Q = Pr$,
... two .... $Q = Pr^2$,
... three .... $Q = Pr^3$,

and so on: that is to say, as the number of turns increases in an *arithmetical* progression the mechanical advantage *increases in a geometrical.*

P. Will the same conclusion hold precisely for a cylindrical surface?

T. Yes; in this case the quantity which I have called r has a value, which I might obtain by making the quantity n which occurs in it indefinitely great; the investigation would, however, be troublesome for you to follow, and would not throw any further light upon the mechanical conditions of the problem: I will therefore leave the result in its present form, merely remarking, that what is true for a polygon of as many sides as you please may be concluded to be true of a circle; and therefore it may be concluded, that if we have a cylindrical post round which a rope is wrapped, the mechanical advantage thereby gained increases in a geometrical progression as the number of turns of the rope increases in an arithmetical. Thus, if with the exertion of a force of 6lbs. I am able with one turn of the cord to sustain a pressure of 24lbs., then $r = \frac{6}{24} = \frac{1}{4}$, and two turns of the rope will enable me to sustain 6×4^2 or 96lbs., three turns 6×4^3 or 384lbs., and so on.

EXAMINATION UPON CHAPTER VII.

1. Determine the direction in which the mutual pressure of two smooth surfaces in contact takes place.

2. Define friction, and enunciate its laws as determined by experiment.

3. Investigate the relation of P to W in the case of the lever, taking into account the action of friction.

4. The same for the inclined plane.

5. The same for the screw.

6. Find the greatest angle of elevation of an inclined plane for which it is possible for a lump of a given substance to rest upon it.

7. A heavy particle is placed upon the exterior of a rough sphere; find the limits within which equilibrium is possible.

8. Given the magnitude of the horizontal force (P), which will just support a given weight (W) upon a plane of given inclination (α); determine the coefficient of friction.

9. There is a block of wood which can be just lifted by the combined strength of two men, determine the greatest angle of elevation of a wooden inclined plane upon which it can be supported by one man, ($\mu = \frac{1}{2}$); the man exerting a force parallel to the plane.

10. Prove that in the preceding problem the elevation of the plane, upon which one man can support the weight, is twice as great as that for which the weight would rest by itself.

11. Explain the manner of making experiments concerning the laws of friction.

12. Assuming the law, that when the extent of surface in contact of one body resting on another is given, the friction for the state bordering on motion varies as the pressure; deduce the truth of the other law of friction, namely, that the friction is independent of the extent of surface in contact.

13. What is meant by *friction wheels?* explain their use.

CHAPTER VIII.

PROBLEMS.

In the present chapter we shall give a number of problems, of such a nature as to be capable of solution by means of the principles already laid down. Some of the problems will be solved by way of illustration, in some hints towards solution will be given, and some will be left entirely to the ingenuity of the student.

The following short collection of hints and rules for the general method of treating problems may probably be found useful.

1. Draw a figure of the system as accurately as possible, representing by arrows the directions of the various forces; all forces of which the magnitude is not known to be denoted by symbols P, Q, R, &c.

2. If the system contains more than one body, the action and reaction between the various bodies at their points of contact must be considered; the action and reaction between two bodies will be always equal in magnitude and opposite in direction, and if the bodies be smooth the action and reaction will take place in the line of the common normal at the point of contact. When these forces have been taken into account, the equilibrium of each component body of the system may be considered separately.

3. Resolve the forces acting upon each body of the system in two directions at right angles to each other, and equate each result to zero: take moments about any point for each body, and equate the result to zero.

In the resolution of the forces, it is not generally of any serious consequence what directions of resolution are

chosen; but in taking moments it is desirable to choose the point about which they are estimated, in such a manner as to give the simplest results; thus, if the directions of two or more forces pass through a point, it is generally desirable to take the moments with reference to that point.

4. Count all the mechanical equations thus produced, and count the unknown quantities both mechanical and geometrical; if the number of unknown quantities exceed that of the equations, the deficiency must be supplied by geometrical equations, that is, by equations expressing necessary geometrical relations amongst the parts of the system.

5. If the surfaces of bodies which act upon each other be rough, the solution of a problem is in general indeterminate except for the state bordering on motion, that is, for the position in which the bodies are on the point of sliding upon each other. In this limiting case, the friction acts in the direction opposite to that in which motion would take place, and is proportional to the mutual normal pressure.

6. The process which has been here described may not unfrequently be abbreviated by artifices peculiar to individual problems. Especially it may be noticed, that when there are only three forces acting upon a body, the position of equilibrium may generally be simply investigated by making the construction of the figure satisfy the necessary condition of the directions of three forces all passing through the same point. The position of equilibrium may also be investigated, upon the principle of the centre of gravity being in either the highest or the lowest position possible; but this method in the greater number of cases requires the aid of the Differential Calculus.

7. The student must not be surprised if the solution of a problem lead to an equation, either algebraical or *trigonometrical*, which he is unable to solve; indeed it not

unfrequently happens, that a problem of apparent mechanical simplicity leads to such a result. This usually arises from the fact of the mode of solution adopted necessarily involving other cognate problems, and therefore giving the solutions of them as well as of that with which we are engaged.

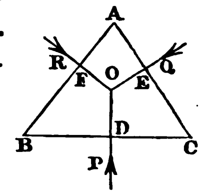
8. It may be remarked, as a matter worthy of the student's attention, that the thorough comprehension of one problem, solved by himself, will do more to assist him in understanding the principles of Statics than the half-comprehension of many. On this account it is recommended to him, when he has solved a problem in one way, to vary his method of proceeding and solve the problem again. Thus he may assume new directions of resolution, and a new point with respect to which to take moments; or he may endeavour to abbreviate the solution by geometrical construction; or he may vary the circumstances of the problem in some of its details. Examples of this will be found in the collection of problems following.

9. In making use of the collection of problems the student is recommended to consider well, and to attempt to solve those problems of which the solution is given in full, before he examines the solutions. The solved problems will be, comparatively speaking, of little profit to him, unless used with this caution.

With these hints the student may enter upon the consideration of the problems which follow, and which of course might be indefinitely multiplied. They will be found to be not exclusively problems concerning the position and conditions of equilibrium of rigid bodies, but amongst them are problems concerning the equilibrium of a particle and the properties of the centre of gravity, and some concerning the equilibrium of a string.

I. Three forces act upon the sides of a triangular board, and in directions perpendicular to the sides: to prove that for equilibrium the forces must be proportional to the sides upon which they act.

Let ABC be the triangle; P, Q, R the forces acting at the points D, E, F respectively; produce the directions of P and Q to meet in O, then FO must be the direction of R, otherwise there could not be equilibrium.



Then by a theorem already proved, p. 80,

$$P : Q : R :: \sin EOF : \sin FOD : \sin DOE,$$
$$:: \sin A : \sin B : \sin C,$$
$$:: BC : CA : AB.$$

II. Treat the preceding problem by resolving the forces and taking moments. Shew that if $BD = x$, $CE = y$, $AF = z$, this method of solution leads to the conclusion,

$$x \sin A + y \sin B + z \sin C = \frac{a^2 + b^2 + c^2}{abc} \times \text{area of triangle};$$

and explain the meaning of this condition.

III. If θ be the angular distance of a body from the lowest point of a circular arc in a vertical plane, the force of gravity in the direction of the arc : that in the direction of the chord :: $2 \cos \frac{\theta}{2} : 1$.

IV. A straight lever of uniform thickness, the length and weight of which are given, has two weights P and Q attached to its extremities, and is maintained partly by a fulcrum at a given point, and partly by another fulcrum on which it presses with a given force; required the position of this latter fulcrum.

V. A beam, 30 feet long, balances about a point at one-third of its length from the thicker end; but when a weight of 10 lbs. is suspended from the smaller end, the fulcrum must be moved 2 feet towards it in order to maintain equilibrium. Find the weight of the beam.

VI. A heavy body is to be conveyed to the top of a rough inclined plane, the angle of inclination being α; prove that if the coefficient of friction be greater than $\dfrac{\sin\left(45^\circ - \frac{\alpha}{2}\right)}{\sin\left(45^\circ + \frac{\alpha}{2}\right)}$, it will be easier to lift the body than to drag it up by means of a cord parallel to the plane.

VII. If G be the centre of gravity of the triangle ABC, then

$$AB^2 + AC^2 + BC^2 = 3\{GA^2 + GB^2 + GC^2\}.$$

VIII. CA, CB are the arms of a bent lever; G is the centre of gravity of the lever; prove that

$$CG^2 = (CA - CB)^2 + \frac{4CA^2 . CB^2}{(CA+CB)^2} \cos^2 \frac{ACB}{2}.$$

IX. Two weights P and Q, connected by a string of given length, balance each other upon the surface of a sphere. Required the position of equilibrium.

X. If three parallel forces, acting at the angular points A, B, C of a triangle, are respectively proportional to the opposite sides a, b, c; prove that the centre of the parallel forces is a point, the distances of which from A, B, C, are respectively,

$$\frac{2bc}{a+b+c} \cos \frac{A}{2}, \quad \frac{2ca}{a+b+c} \cos \frac{B}{2}, \quad \frac{2ab}{a+b+c} \cos \frac{C}{2}.$$

XI. Two forces F and F', acting in the diagonals of a parallelogram, keep it at rest in such a position that one of its edges is horizontal; shew that $F \sec \alpha = F' \sec \alpha' = W \cos (\alpha + \alpha')$, where W is the weight of the parallelogram, α and α' the angles between its diagonals and the horizontal side.

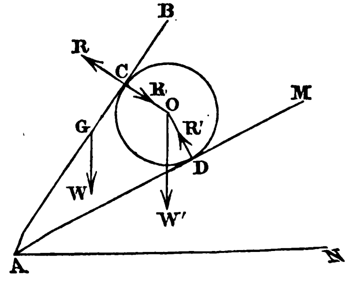
XII. A uniform beam AB, of given length and weight, rests with one end on a given inclined plane, and the other attached to a string AFP passing over a pully F at a given point of the inclined plane. Knowing the weight P fixed to the other end of the string, find the position in which the beam rests.

XIII. Find the limits of possibility of the preceding problem.

XIV. Also solve the problem for the particular case in which the string AF is horizontal.

XV. By means of tables obtain a numerical result for the angle between the beam and the plane, when the inclination of the plane is 45^0, the weight of the beam 4lbs., and $P = 3$lbs.

XVI. A smooth sphere of which the centre is O, rests upon an inclined plane AM at D, and is kept in equilibrium by the uniform beam AB which moving about a hinge at A presses upon the sphere at C. Required the conditions of equilibrium.



Let $\alpha = MAN$, the angle of the plane; W, W' be the weights of the

beam and sphere respectively. At the point C there will be an action of the sphere upon the beam, and an equal and opposite action of the beam upon the sphere; call this R, its direction will be perpendicular to the beam and through the centre of the sphere. At D there will be an action of the plane upon the sphere, R' suppose, its direction will be DO. The beam will be kept in equilibrium by the forces R, W acting at G the middle point of AB, and a pressure at A which we need not consider; the sphere, by the three forces W' at O, R and R'.

Let $r =$ the radius of the sphere; $AC = AD = x$; $CAD = \theta$;

Then for the beam, taking moments about A,

$$Rx = Wa \cos(\theta + \alpha) \qquad (1).$$

For the sphere,

$$R \sin(\theta + \alpha) = R' \sin \alpha \qquad (2),$$

$$R \cos(\theta + \alpha) + W' = R' \cos \alpha \qquad (3).$$

And we have the geometrical relation,

$$a = x \tan \frac{\theta}{2}, \qquad (4).$$

Multiplying (2) by $\cos \alpha$ and (3) by $\sin \alpha$ and subtracting, we have,

$$R\{\sin(\theta + \alpha)\cos \alpha - \cos(\theta + \alpha)\sin \alpha\} - W' \sin \alpha = 0,$$

$$\text{or } R \sin \theta = W' \sin \alpha;$$

$\therefore$ combining this equation with (1), there results,

$$\frac{x}{\sin \theta} = \frac{Wa \cos(\theta + \alpha)}{W' \sin \alpha},$$

$$\text{or } \frac{W \cos(\theta + \alpha)}{W' \sin \alpha} = \frac{\cot \frac{\theta}{2}}{\sin \theta} = \frac{\cot \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} = \frac{1}{2 \sin^2 \frac{\theta}{2}};$$

$$\therefore 2 \frac{W}{W'} \cos(\theta + \alpha) \sin^2 \frac{\theta}{2} = \sin \alpha,$$

$$\text{or } \sin \alpha \operatorname{cosec}^2 \frac{\theta}{2} = (2 \cos \alpha \cos \theta - 2 \sin \alpha \sin \theta) \frac{W}{W'},$$

$$\frac{W'}{W}\left(1 + \cot^2 \frac{\theta}{2}\right) = 2 \cot \alpha \left(\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}\right) - 4 \sin \frac{\theta}{2} \cos \frac{\theta}{2};$$

$$\therefore \frac{W'}{W}\left(1 + \cot^2 \frac{\theta}{2}\right)^2 = 2 \cot \alpha \left(\cot^2 \frac{\theta}{2} - 1\right) - 4 \cot \frac{\theta}{2},$$

$$\text{or } \frac{W'}{W}\left(\frac{x^2}{a^2} + 1\right)^2 + 4 \frac{x}{a} = 2 \cot \alpha \left(\frac{x^2}{a^2} - 1\right).$$

This is the simplest form to which the equation can be reduced; it has been already remarked, that not unfrequently statical problems of no great degree of complication lead (as in this case) to equations which we are not able to solve.

If the position of equilibrium be assigned, that is, if θ or x be given, the equation which we have obtained gives us the ratio of the weight of the sphere to that of the beam, for we have

$$\frac{W'}{W}=\frac{2\cot\alpha\,(x^2-a^2)-4ax}{(x^2+a^2)^2}.$$

From this equation we can obtain a limit of the possibility of the problem; for $\frac{W'}{W}$ must be positive, therefore we must have

$$2\cot\alpha\,(x^2-a^2)-4ax \text{ positive,}$$

$$\text{or } x^2-a^2>2ax\tan\alpha,$$

$$\text{or } \tan\alpha<\frac{x^2-a^2}{2ax}.$$

Suppose, for instance, that $x=2a$, then

$$\tan\alpha \text{ must be } <\tfrac{3}{4}<.75,$$

$$\text{or } \alpha<36^\circ.\,52';$$

$$\text{and then } \frac{W'}{W}=\frac{2\cot\alpha\times3-8}{25}.$$

Thus if $\alpha=30^\circ$, $\cot\alpha=\sqrt{3}$,

$$\text{and } \frac{W'}{W}=\frac{6\sqrt{3}-8}{25}=\frac{10.39-8}{25}=\frac{1}{10} \text{ approximately;}$$

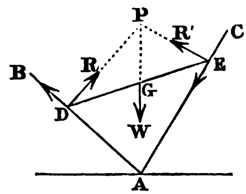
that is, if the inclination of the plane be 30°, and the distance of either point of contact of the sphere from the hinge equal to the diameter, then for equilibrium the beam must be about 10 times as heavy as the sphere.

In the particular case of the weights of the sphere and beam being equal, the biquadratic upon which the problem depends is reducible to a cubic: this the student may prove.

XVII. A uniform rod, 50 feet long, is cut into two pieces of 18 and 32 feet respectively, and these are placed upon a smooth horizontal plane with their lower extremities pressing against each other, and the upper extremities leaning against two smooth vertical planes; also the two rods are at right angles to each other; find the distance between the walls.

XVIII. A uniform beam rests with its extremities upon two given inclined planes; to find the position of equilibrium.

Let AB, AC be the inclined planes, α, β, their respective inclinations; DE the beam, G its middle point or centre of gravity, W its weight, $2a$ its length; R, R' the actions of the two planes at D and E, which will be perpendicular to AB and AC.



Then if θ be the angle which DE makes with the horizon, we shall have by resolving horizontally and vertically and taking moments about G,

$$R \sin \alpha - R' \sin \beta = 0 \text{..............................(1)}$$

$$R \cos \alpha + R' \cos \beta = W \text{ ....(2)}$$

$$Ra \cos (\alpha + \theta) - R' a \cos (\beta - \theta) = 0 \text{ ....(3).}$$

from (1) and (3)

$$\frac{\cos (\alpha + \theta)}{\sin \alpha} = \frac{\cos (\beta - \theta)}{\sin \beta};$$

or, $\quad \cot \alpha \cos \theta - \sin \theta = \cot \beta \cos \theta + \sin \theta;$

or, $\quad 2 \tan \theta = \cot \alpha - \cot \beta.$

XIX. Let us solve the preceding problem by a different method. The directions of R and R' if produced must meet in a certain point P, and the direction of W produced must pass through this same point. Suppose them to do so; then in the triangle PDG, $GPD = \alpha$, (for it is the inclination of DP to the vertical, which must be the same as that of AB to the horizon,) $PDG = 90^\circ - ADG = 90^\circ - \alpha - \theta$,

$$\therefore \frac{PG}{DG} = \frac{\sin PDG}{\sin GPD} = \frac{\cos (\alpha + \theta)}{\sin \alpha};$$

in like manner, from the triangle PEG,

$$\frac{PG}{EG} = \frac{\cos (\beta - \theta)}{\sin \beta};$$

but $DG = EG$;

$$\therefore \frac{\cos (\alpha + \theta)}{\sin \alpha} = \frac{\cos (\beta - \theta)}{\sin \beta}, \text{ as before.}$$

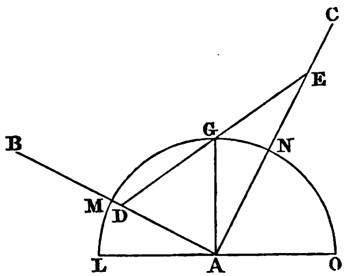
XX. In the particular case in which the angle between the two planes is a right angle the result assumes a simpler form, and moreover gives rise to a remark upon a still different method of solving the problem. In this case, $\beta = 90^\circ - \alpha$, and our equation becomes

$$2 \tan \theta = \cot \alpha - \tan \alpha = \frac{1}{\tan \alpha} - \tan \alpha = \frac{1 - \tan^2 \alpha}{\tan \alpha};$$

$$\therefore \tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{1}{\tan\theta} = \cot\theta = \tan(90^0 - \theta);$$

$$\text{or,} \quad 2\alpha = 90^\circ - \theta, \qquad \text{or,} \quad \theta = 90^0 - 2\alpha.$$

Now suppose in this case that we make the beam DE to assume all possible positions while its extremities move upon the planes AB, AC; then it is well known (and is easily proved) that the middle point G will trace out an arc MGN of a circle LGO of which the centre is A, and the diameter DE, (LO is taken as the horizontal line). Then, since we may regard the weight of the beam as collected at G, the problem is precisely the same as that of finding the position of equilibrium of a particle upon the surface of the semicircle LGO. It is manifest that this position will be the highest point of the circle; therefore draw AG vertical, and the intersection of this line with the circle will fix the position of G; through G draw DGE so as to be bisected in G, (which may be done in several ways,) then DGE is the required position of the beam. And we have,



$$GAD = GDA = \alpha + \theta;$$

$$\text{but } GAD = 90^0 - DAL = 90^0 - \alpha;$$

$$\therefore \alpha + \theta = 90^0 - \alpha,$$

$$\text{or } \theta = 90^0 - 2\alpha, \text{ as before.}$$

The problem might be worked out in a similar manner for the more general case, but the locus of G would then be an ellipse instead of a circle, and the solution would not be so simple. This method of solution has the advantage of pointing out the character of the equilibrium of this problem with respect to *stability* (see p. 66); for it is evident that the equilibrium of a particle upon a smooth circle is unstable or only theoretically possible, therefore also the equilibrium of the beam DE is unstable. Practically the influence of friction will make the equilibrium possible, and that within considerably wide limits: let us investigate this case.

XXI. Suppose the extremity D to be on the point of descending, then the friction at D will be in the direction BD, that at E will be in the direction EA, and the equations of the problem will be as follows:

$$R\sin a-\mu R\cos a-R'\sin\beta-\mu R'\cos\beta=0 \qquad (1)$$
$$R\cos a+\mu R\sin a+R'\cos\beta-\mu R'\sin\beta=W \qquad (2)$$
$$Ra\cos(a+\theta)+\mu Ra\sin(a+\theta)-R'a\cos(\beta-\theta)+\mu R'a\sin(\beta-\theta)=0 \quad (3).$$

From (1) and (3) we have

$$\frac{\cos(a+\theta)+\mu\sin(a+\theta)}{\sin a-\mu\cos a}=\frac{\cos(\beta-\theta)-\mu\sin(\beta-\theta)}{\sin\beta+\mu\cos\beta}.$$

Let us simplify this equation by putting $\mu=\tan f$, then

$$\frac{\cos(a+\theta-f)}{\sin(a-f)}=\frac{\cos(\beta-\theta+f)}{\sin(\beta+f)};$$

$$\text{and } 2\tan\theta=\cot(a-f)-\cot(\beta+f).$$

This gives us the value of θ when D is on the point of descending; if, on the other hand, D be on the point of ascending, we have only to change the sign of f, and calling the value of θ for this case θ', we have

$$2\tan\theta'=\cot(a+f)-\cot(\beta-f).$$

In the particular case in which $\beta=90^0-a$,

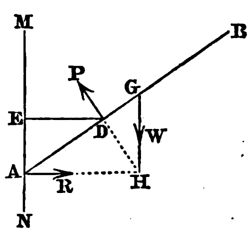
$$2\tan\theta=\cot(a-f)-\tan(a-f),$$
$$\text{and } \theta=90^0-2a+2f,$$
$$\text{similarly, } \theta'=90^0-2a-2f,$$
$$\therefore\ \theta-\theta'=4f,$$
$$\text{or } f=\frac{\theta-\theta'}{4};$$

that is, the subsidiary angle f which we have introduced is one fourth of the angle between the two extreme positions of the beam.

XXII. Vary the preceding problem by taking one of the planes vertical; and let the planes be (1) smooth, (2) rough.

XXIII. Vary it again by considering it as in XXI, except that one plane shall be rough and the other smooth.

XXIV. A uniform beam rests upon a prop, with one extremity in contact with a smooth vertical wall. To find the conditions of equilibrium.



Let AB be the beam; G its middle point, or centre of gravity, W its weight, $2a$ its length, MN the wall, R the pressure of the wall upon the beam, which will be horizontal. D the prop, c its distance DE from the wall, P the pressure of the prop on the beam, which will be in a direction perpendicular to AB. *Also let* θ be the angle which AB makes *with the horizon*, or $\theta=ADE$.

Then resolving horizontally, and vertically, and taking moments about A,

$$R - P\sin\theta = 0 \;\ldots\ldots\; (1)$$

$$W - P\cos\theta = 0 \;\ldots\ldots\; (2)$$

$$Pc\sec\theta - Wa\cos\theta = 0 \;\ldots\ldots\; (3)$$

from (2) and (3) $\quad c\sec\theta = a\cos^2\theta,$

$$\text{or } \cos^3\theta = \frac{c}{a},$$

$$\therefore\; \cos\theta = \left(\frac{c}{a}\right)^{\frac{1}{3}},$$

which determines θ. It appears from the result that c must be less than a, as is represented in the figure, and as must manifestly be the case.

XXV. We may solve this problem otherwise. In the triangle AED, the sides AE, ED, DA are respectively perpendicular to the directions of the forces R, W, P; hence they are in the same proportion as those forces, that is,

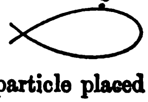
$$R : W : P :: AE : ED : DA,$$

$$:: \sin\theta : \cos\theta : 1,$$

This proportion is equivalent to equations (1) and (2). But we must obtain another equation in order to solve the problem; this is done from the consideration that the directions of three forces must meet in one point; let them meet in H. Then

$$\cos\theta = \frac{AD}{AH} = \frac{c\sec\theta}{a\cos\theta}, \text{ or } \cos^3\theta = \frac{c}{a} \text{ as before.}$$

This problem cannot be conveniently worked out upon the principle of making the centre of gravity of the beam assume its lowest or highest position without the use of the Differential Calculus. By drawing the beam, however, in several different positions, it is easy to satisfy ourselves that the curve which will be traced out by the centre of gravity G will be of the form represented in the figure, and that the position of equilibrium which we have determined corresponds to the *highest* point Q of this curve.



Consequently the equilibrium is analogous to that of a particle placed at the highest point of a curve, from which it will, if disturbed, immediately descend; that is, the equilibrium is *unstable*.

XXVI. Vary the preceding problem by taking the plane MN inclined at an angle α to the horizon.

If ϕ be the angle between the beam and MN, the result is

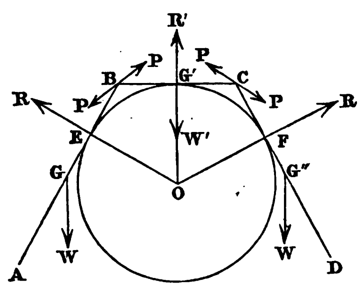
$$\sin^2\phi\cos(\alpha-\phi)=\frac{c}{a}\sin\alpha.$$

XXVII. Take the problem as before, with the exception of supposing a weight W' suspended from B.

XXVIII. If the wall be rough, find the limiting positions of equilibrium.

XXIX. Three uniform beams of length $2a$, $2b$, and $2a$ respectively, and of equal thickness, are loosely jointed together and suspended symmetrically from a fixed cylinder, of which the axis is horizontal and the radius greater than b, the middle beam resting upon the cylinder. Determine the pressure at the three points of contact; (1) when $2a$ is greater than b; (2) when it is less.

Let AB, BC, CD be the three beams, E, G', F the points of contact, G the middle point of AB. The figure will sufficiently explain the forces which act; it need only be remarked, that the action between the two beams at B is assumed to be one force P acting in a direction determined by the angle θ which it makes with the horizon; we might have assumed a horizontal and vertical force, but have chosen to assume one force in a direction to be determined.



Let r be the radius of the cylinder, $EOG'=\alpha$, then the equations for the beam AB are

$$P\cos\theta-R\sin\alpha=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$P\sin\theta+R\cos\alpha=W \quad\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$Rb-Wa\cos\alpha=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

for the beam BC,

$$R'-2P\sin\theta=W' \quad\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

the other two equations are identical; and those for CD are the same as for AB. And we have the geometrical relation,

$$b=r\tan\frac{\alpha}{2} \quad\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

From (3),

$$R = \frac{Wa}{b}\cos\alpha = \frac{Wa}{b}\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = \frac{Wa}{b}\frac{r^2-b^2}{r^2+b^2}.$$

It will be observed that the problem is set with the condition that r is $> b$, and it is manifest that the beams AB, CD could not otherwise be in contact with the cylinder; in the expression just found R, if r be less than b, R becomes negative, which shews that the beam AB could not be in contact unless held by a string, the tension of which would be a force in the opposite direction to that assigned in the figure to R.

Again, from (4),

$$\begin{aligned} R' &= W' + 2P\sin\theta, \\ &= W' + 2W - 2R\cos\alpha, \quad \text{from (2)}, \\ &= W' + 2W - 2W\frac{a}{b}\cos^2\alpha, \\ &= W' + 2W\left\{1 - \frac{a}{b}\cdot\left(\frac{r^2-b^2}{r^2+b^2}\right)^2\right\}. \end{aligned}$$

Since the beams are of equal thickness we must have

$$W' : W :: b : a;$$

$$\therefore R' = W\left\{\frac{b}{a} + 2 - \frac{2a}{b}\left(\frac{r^2-b^2}{r^2+b^2}\right)^2\right\}.$$

Let us simplify these expressions by supposing that $r = 2b$,

$$\text{then } R = \frac{3a}{5b}W, \quad R' = W\left(\frac{b}{a} + 2 - \frac{18a}{25b}\right).$$

Still further, suppose that $a = 2b$,

$$\text{then } R = \frac{6}{5}W, \quad R' = \left(\frac{1}{2} + 2 - \frac{36}{25}\right)W = \frac{53}{50}W.$$

XXX. The case in which $2a$ is $< b$ we shall leave to the student's own ingenuity, merely remarking that he will find the mechanical equations, with one exception, to be the same, and that he must substitute a new geometrical relation for equation (5). No step can be conveniently taken for the solution of the problem beyond writing down the appropriate equations.

XXXI. Two equal and similar rough beams are fastened together by two of their extremities, so as to include a given angle and together

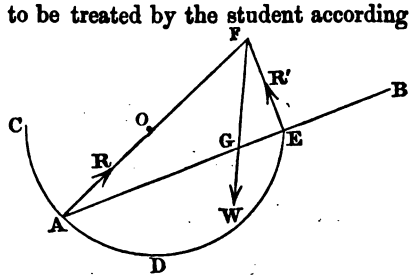
to form one rigid piece; the piece thus formed is balanced upon a fixed rough horizontal cylinder. Determine the limiting positions of equilibrium, the coefficient of friction being $\tan f$.

The student may, if he please, simplify this problem by taking the angle between the beams a right angle, and each of the beams equal in length to the diameter of the cylinder.

XXXII. A person suspended in a balance of which the arms are equal, thrusts his centre of gravity out of the vertical by means of a rod fixed to the further extremity of the beam of the balance, the direction of the rod passing through his centre of gravity: given that the rod and the line from the nearer end of the beam of the balance to his centre of gravity make angles α, β with the vertical, shew that his apparent and true weights are in the ratio $\sin(\alpha+\beta) : \sin(\alpha-\beta)$.

XXXIII. A uniform beam is placed in a fixed smooth hemispherical bowl, the diameter of which is less than the length of the beam, find the position of equilibrium.

We will leave this problem to be treated by the student according to the general method of resolution of forces and of moments, and will insert a geometrical solution.



Let AB be the beam, CDE the bowl, O its centre; draw AOF, and EF perpendicular to AB, to meet in F; these will be the directions of the two pressures R, R', at A and E. From F draw a vertical line, which must pass through G, the middle point of AB and be the line of action of its weight W. Let r be the radius of the sphere, $AB = 2a$, θ the angle which AB makes with the vertical, or $FGE = \theta$.

Then $OAE = OEA = 90° - \theta$. Also since AEF is a right angle, a semicircle described upon AF will pass through E, $\therefore AO = OF$, or $AF = 2r$.

Now from the triangle FAG, we have

$$\frac{AG}{AF} = \frac{\sin AFG}{\sin AGF},$$

$$\text{or } \frac{a}{2r} = \frac{\sin(\theta - 90° + \theta)}{\sin\theta} = -\frac{\cos 2\theta}{\sin\theta};$$

$$\therefore \cos 2\theta + \frac{a}{2r}\sin\theta = 0,$$

$$\text{or } \sin^2\theta - \frac{a}{4r}\sin\theta - \tfrac{1}{2} = 0,$$

a quadratic for determining θ.

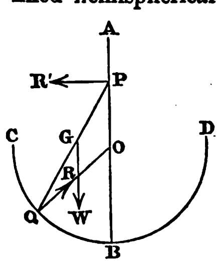
The student may perhaps be puzzled by obtaining a quadratic, which must have of necessity two roots, when apparently one answer only was required to the problem: by solving the equation, (or by observing the sign of its last term,) he will perceive that one root of the equation is positive and the other negative, and since the angle which he requires is manifestly less than 180° he will know that the positive root is that which he seeks. The full explanation however of the existence of this negative root would lead us into difficult considerations, which are better for the present omitted.

With regard to the stability of this beam the student will have no difficulty in satisfying himself, by considering the form of the curve which G must necessarily describe if the beam be made to assume different positions, that the equilibrium is stable.

XXXIV. Consider the preceding problem, taking account of friction.

XXXV. An aperture is made at the extremity of a horizontal diameter of a fixed hollow sphere, and a rod without weight and of greater length than the diameter of the sphere inserted; given the length of the rod, determine the conditions of equilibrium when it is acted upon by a given horizontal force, at the extremity which lies without the sphere.

XXXVI. AOB is a fixed vertical rod in a fixed hemispherical bowl CBD, of which the centre is O. PQ is a uniform rod resting upon the bowl at Q and upon AOB at P: to find the position of equilibrium.



Join QO, this will be the direction of the pressure of the surface of the bowl upon the rod at Q, which call R. The pressure R' of the rod AOB upon PQ at P will be horizontal. The weight W of PQ will act parallel to AOB at the middle point G of PQ. Let $QPB=\theta$, $QOB=\phi$, $BO=r$, and $PQ=2a$.

Then resolving horizontally and vertically, and taking moments about G, we have

$$R' - R\sin\phi = 0 \ldots\ldots\ldots (1)$$

$$W - R\cos\phi = 0 \ldots\ldots\ldots (2)$$

$$R'a\cos\theta - Ra\sin(\phi - \theta) = 0 \ldots (3)$$

And we have the geometrical relation,

$$\frac{2a}{r} = \frac{\sin\phi}{\sin\theta} \ldots\ldots\ldots\ldots (4)$$

Now from (1) and (3) there results

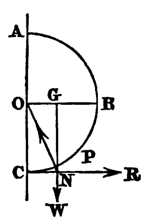
$$\sin\phi\cos\theta = \sin(\phi - \theta)$$
$$= \sin\phi\cos\theta - \sin\theta\cos\phi;$$
$$\therefore \sin\theta\cos\phi = 0.$$

Hence either $\theta = 0$, or $\phi = 90°$; the latter is not possible, because (2) would give us $W = 0$; hence $\theta = 0$ is the only solution, that is, the rod PQ must stand vertically, in which case it is manifest that there will be equilibrium.

Consequently there is no such position of equilibrium as that which we have represented in the figure; and a little consideration would have pointed this out to us at first, for it is evident that the directions of the three forces W, R, R' cannot pass through the same point, which is an essential condition of equilibrium.

XXXVII. A ladder being placed against a perfectly smooth wall, find the smallest angle of elevation for which equilibrium is possible, the coefficient of friction between the ladder and ground being given.

XXXVIII. A hemisphere ABC being placed in contact with a smooth vertical wall, as in the figure; to find the nearest point to C at which a smooth prop being placed, the hemisphere will be at rest.



Let O be the centre of the sphere; draw OGB horizontal, then it is evident that the centre of gravity of the hemisphere will be somewhere in OB: let it be G; the exact position of G is a matter of indifference so far as the method of investigation is concerned, but we may as well assume that which can be proved, namely, that $OG : OB :: 3 : 8$. Let W be the weight of the hemisphere which acts at G.

Again, if there be a prop at any point P, it is plain that the hemisphere cannot fall without twisting so as to separate itself from the wall; just before the final separation takes place, the lowest point C will be the only point in contact with the wall: hence we may suppose the action of the wall to produce a force R at the point C, and in the direction perpendicular to AC, i. e. in the horizontal direction.

Let the directions of W and R intersect in N. Then the third force arising from the pressure of the prop must also act through N; but this force must also act through O, since it must be normal to the surface of the sphere; hence, joining NO, this must be the direction of the force, and the point in which NO meets the hemisphere will be the point required.

If we denote by θ the angle CON, then $\tan\theta = \frac{CN}{OC} = \frac{3}{8}$, or $\theta = 20^\circ 33'$: and the angle θ thus found will entirely fix the position of the point required.

We have solved this problem by a geometrical construction, as being the neatest method of treatment. If however we had taken P as the point, and denoted COP by θ, and the pressure at P by R', we should have had the equations

$$R' \cos\theta = W \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$R' \sin\theta = R \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$R \,.\, OC = W \,.\, OG \ldots\ldots\ldots\ldots\ldots\ldots (3);$$

(1) and (2) give us,

$$\tan\theta = \frac{R}{W},$$

and from (3)

$$\frac{R}{W} = \frac{OG}{OC} = \frac{3}{8},$$

$$\therefore \tan\theta = \frac{3}{8} \text{ as before.}$$

XXXIX. Consider the preceding problem on the supposition of the wall being rough.

XL. If the wall be smooth and the prop be not at the nearest point possible to C, but at any other point P such that $COP = \alpha$, determine the pressure upon the wall and prop respectively.

XLI. A cylinder lies upon two equal cylinders, all in contact, and having their axes parallel: and the lower cylinder rests on a horizontal plane: $\tan f$, $\tan f'$ are the coefficients of friction respectively between the cylinders, and between the cylinders and the plane. Supposing the points of contact all to slip at the same instant, find f and f'.

Let W be the weight of the upper cylinder, W' that of either of the lower ones, R the action between two cylinders, R' between either of the lower cylinders and the plane; a the angle which the line joining the centres of the upper and either of the lower cylinders makes with the vertical. (The student can supply the figure.) Then the equations of the problem will be as follows.

For the upper cylinder,

$$2R\cos a + 2R\tan f\sin a = W,$$

$$\text{or } R\cos(a-f) = \frac{W\cos f}{2} \text{ .... (1).}$$

For either of the lower cylinders,

$$R\sin(a-f) = \cos f\tan f' R' \text{ .... (2),}$$

$$R\cos(a-f) = (R'-W')\cos f \text{ .... (3),}$$

$$R\tan f = R'\tan f' \text{ .... (4).}$$

From these four equations we can eliminate R and R', and find f and f'. From (2) and (4), we have

$$\sin(a-f) = \sin f;$$

$$\therefore\ a-f=f, \text{ or } f=\frac{a}{2}.$$

Again, from (1) and (3),

$$\frac{W}{2} = R'-W', \text{ or } R' = W' + \frac{W}{2},$$

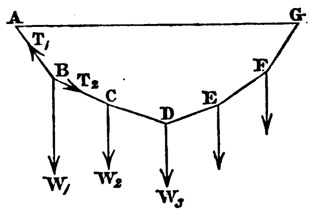
from (1) and (2)

$$\tan(a-f) = \frac{2R'\tan f'}{W},$$

$$\text{or } \tan\frac{a}{2} = \frac{2W'+W}{W}\tan f';$$

$$\therefore\ \tan f' = \frac{W}{2W'+W}\tan\frac{a}{2}.$$

XLII. A number of weights W_1, W_2, W_3, ... are connected by strings of given length, the extremities being attached to two points in the same horizontal line, so as to allow the weights to hang in a festoon below; to find the conditions of equilibrium.—*The funicular polygon.*



Let B, C, D,....be the weights; A, G the points of suspension; $AB = a_1$, $BC = a_2$, &c. $AG = c$; also let the angles which AB, BC,....respectively make with the vertical be $\theta_1\ \theta_2$....; lastly, let $T_1\ T_2$ ...... be the tensions of the strings.

Then the weight W_1 is kept in equilibrium by its own weight acting vertically, T_1 acting in the direction BA, and T_2 in the direction BC. Hence resolving horizontally and vertically we have,

$$T_1 \sin\theta_1 = T_2 \sin\theta_2 \text{ .... } (1)$$

$$T_1 \cos\theta_1 - T_2 \cos\theta_2 = W_1 \text{ .... } (2)$$

We shall have two similar equations for each weight; suppose there are n of them, then we shall have $2n$ equations. Let us consider how many unknown quantities there will be. There are $n+1$ strings, therefore there will be $n+1$ unknown forces $T_1\ T_2$ ...... T_{n+1}, and $n+1$ unknown angles $\theta_1\ \theta_2$ ...... θ_{n+1}; on the whole then there will be $2n+2$ unknown quantities; but there are only $2n$ mechanical equations, therefore we must have two geometrical. One of these will be the following, which arises from the fact of AG being given,

$$a_1 \sin\theta_1 + a_2 \sin\theta_2 + \text{......} + a_{n+1} \sin\theta_{n+1} = c. \quad \text{(A)}$$

The other is obtained thus; let D be the lowest point of the festoon, and let W_p be the weight which hangs at D, then the vertical distance of D from AG is

$$a_1 \cos\theta_1 + a_2 \cos\theta_2 + \text{\&c.} + a_p \cos\theta_p;$$

but the same vertical distance is also equal to

$$a_{p+1} \cos\theta_{p+1} + \text{......} + a_{n+1} \cos\theta_{n+1};$$

$$\therefore\ a_1 \cos\theta_1 + \text{......} + a_p \cos\theta_p = a_{p+1} \cos\theta_{p+1} + \text{......} + a_{n+1} \cos\theta_{n+1}. \quad \text{(B)}$$

Thus we have obtained the $2n+2$ equations required.

To complete the solution of the problem so far as it can be completed, let us write down equation (2) and all those of the same

class: and to simplify the problem we will consider all the weights equal, then

$$T_1 \cos\theta_1 - T_2 \cos\theta_2 = W,$$
$$T_2 \cos\theta_2 - T_3 \cos\theta_3 = W,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$T_n \cos\theta_n - T_{n+1} \cos\theta_{n+1} = W;$$

subtracting the second of these equations from the first, we have

$$T_1 \cos\theta_1 - 2\,T_2 \cos\theta_2 + T_3 \cos\theta_3 = 0.$$

But by (1) $T_1 \sin\theta_1 = T_2 \sin\theta_2$,

and in like manner, $T_2 \sin\theta_2 = T_3 \sin\theta_3$,

$$\therefore\ T_1 \sin\theta_1 = T_2 \sin\theta_2 = T_3 \sin\theta_3 = \lambda \text{ suppose};$$

$$\therefore\ T_1 = \frac{\lambda}{\sin\theta_1},\quad T_2 = \frac{\lambda}{\sin\theta_2},\quad T_3 = \frac{\lambda}{\sin\theta_3}.$$

Hence our equation becomes

$$\frac{\lambda\cos\theta_1}{\sin\theta_1} - 2\frac{\lambda\cos\theta_2}{\sin\theta_2} + \frac{\lambda\cos\theta_3}{\sin\theta_3} = 0,$$

$$\text{or}\quad \cot\theta_1 + \cot\theta_3 = 2\cot\theta_2.$$

The angles θ_1, θ_2, θ_3, therefore, and in like manner all the following angles, are such that their cotangents form an arithmetical progression. In this arithmetical progression it is evident that the terms must be decreasing, (since the angles increase, and therefore their cotangents diminish;) hence the terms will at length become negative, that is, the angle will be greater than a right angle, and these will correspond to the strings between the lowest weight and the point G.

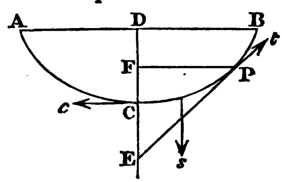
The set of equations similar to (1) express that the horizontal tension of all the strings is the same; a result which might have been anticipated. And this same thing is true of a chain or cord suspended from its two extremities, and so forming what is called a *catenary;* that is to say, the horizontal tension is the same at all points of the chain or cord.

XLIII. Two equal weights are suspended from two points in the same horizontal line, which are two feet apart, by means of three threads which are respectively 1, 1 and 2 feet long; prove that the three acute angles which the threads make with the horizontal line are together equal to two right angles.

XLIV. If in the preceding problem the connecting threads be each one foot long, determine the angles which they make with the horizon, and their tensions.

XLV. Having alluded to the form assumed by a cord or chain suspended from its two extremities, it may be as well to make one or two more remarks upon the same subject. The form of the curve we shall not be able to investigate, without recourse to a higher portion of the pure mathematics than we suppose the student to have at his command; nevertheless certain properties of the curve may be investigated without difficulty.

Let ACB be a uniform cord, hanging from two points A B in the same horizontal line; D the middle point between A and B; DC a vertical line. Then it is evident, that the portion AC of the cord on one side of DC will be precisely similar to the portion BC on the other side, and C will be the lowest point.



Now how are we to apply our principles of equilibrium, which have been investigated for a *particle* and for a *rigid body*, to the case of a *cord* which is neither the one nor the other? The method adopted is as follows.

Let P be any point in the cord, and let the length of $CP = s$. Then since the cord is at rest, it will not affect the equilibrium if we *suppose* the portion CP to become rigid; suppose, for instance, CP to be impregnated with some fluid which solidifies, so that CP hangs like a wire with cords at its extremities. The rigid piece CP will be held in equilibrium by the vertical force of its own weight, which will be proportional to s and will act through its centre of gravity, and by the tensions of the cord at C and P; now these tensions will be in the directions of the tangents of the curve at those points, therefore the tension at C will be evidently horizontal, and that at P will be in the direction of EP, suppose. We have already explained that the horizontal tension will be the same at all points of the *catenary*, therefore the tension at C which is wholly horizontal will be equal to this constant horizontal tension; let c be the length of cord the weight of which would be equal to this tension, then as the weight of CP is to s, so is the tension at C to c. In like manner, denote the unknown tension at P by t, where t is in fact the length of cord whose weight would produce that tension.

Now draw PF horizontal, that is, perpendicular to DCE. Then CP is kept in equilibrium by the three forces s, c, t which act in directions respectively parallel to FE, PF, EP; therefore by the Triangle of Forces,

$$s : c : t :: FE : PF : EP.$$

Let $FEP = \theta$, then

$$\frac{s}{c} = \frac{FE}{PF} = \cot\theta \;\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\text{and } \frac{t}{c} = \frac{EP}{PF} = \operatorname{cosec}\theta, \text{ or } t^2 = c^2 + s^2 \ldots\ldots (2).$$

We cannot proceed any further in the solution; but it will be seen that we have obtained two important results.

For (1) gives us the law according to which the direction of the tangent of the catenary changes as we proceed from the lowest point; and it will be interesting to notice how this result may be obtained from the corresponding equation in the investigation of the funicular polygon.

Now we may regard a uniform cord as a series of equal weights connected by strings of indefinitely small length, and then the direction of the string joining any two weights will be the direction of the tangent at the corresponding point of the catenary. Let then the cord s be divided into any number (p) of equal parts; then the weight of each portion will be measured by $\frac{s}{p}$. Now resuming equations (1) and (2) of the funicular polygon (p. 193), we have

$$T_1 \sin\theta_1 = T_2 \sin\theta_2 = c,$$

$$T_1 \cos\theta_1 - T_2 \cos\theta_2 = \frac{s}{p};$$

$$\therefore\; c\cot\theta_1 - c\cot\theta_2 = \frac{s}{p},$$

$$\text{or, } \cot\theta_1 - \cot\theta_2 = \frac{1}{p}\cdot\frac{s}{c},$$

$$\text{in like manner, } \cot\theta_2 - \cot\theta_3 = \frac{1}{p}\cdot\frac{s}{c},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots$$

$$\text{and } \cot\theta_p - \cot\theta_{p+1} = \frac{1}{p}\cdot\frac{s}{c};$$

therefore, adding all these equations together,

$$\cot\theta_1 - \cot\theta_{p+1} = p \times \frac{1}{p}\cdot\frac{s}{c} = \frac{s}{c}.$$

But at the lowest point of the catenary $\theta_{p+1} = 90^\circ$, or $\cot\theta_{p+1} = 0$, and θ_1 is the angle which we have called θ in the investigation of the catenary;

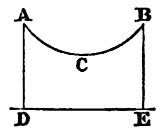
$$\therefore \cot\theta = \frac{s}{c} \text{ as before.}$$

Again, equation (2) gives us the law according to which the tension of the cord in the catenary varies from point to point. This may be in like manner deduced from the equations (1) of the funicular polygon.

XLVI. The following is an example of the application of the pre ceding investigation.

A chain of given weight is suspended from two equal vertical posts of given height, and the angle which the chain makes with either post at the point of suspension is observed. To find the moment of the force which tends to overturn one of the posts.

If h be the height of one of the posts, then, according to our previous notation, ch is the moment required. To determine c, we observe, that if W be the whole weight of the chain, and α the angle which the chain makes with either post, then



$$\frac{W}{c} = \cot\alpha,$$

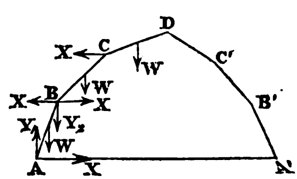
$$\therefore c = W\tan\alpha,$$

and $Wh\tan\alpha$ is the moment required.

It may be remarked here, that although a cord may be stretched so as to be for all practical purposes horizontal, as in the case of a pianoforte string, yet it never can be so accurately. In the preceding investigation α cannot be 90° unless either $W = 0$ or $c = \infty$.

XLVII. Analogous to the problem of the funicular polygon is that of the conditions of equilibrium of a system of beams, resting one upon another so as to form a kind of arch. The essential distinction between the two problems is, that in the case of the beams we have a series of equations of *moments*, which do not occur in the other.

To simplify the problem we will suppose the beams to be all equal and uniform, and that there are an even number of them. Then the arrangement will be such as is represented in the figure.



9—3

Let us consider the forces which act. Upon each beam there will be the vertical force W, its weight, acting at its middle point. At each extremity of each beam there must be a force, which it will be convenient to consider as resolved into a horizontal and a vertical part; this will be convenient, because it is evident that all the horizontal parts must be the same throughout the system, and we may therefore denote them by one common symbol X: let Y_1, Y_2, Y_3, ... be the upward vertical pressures at the lower extremities of the 1st, 2nd, 3rd, ... beams respectively, reckoning from the lowest; and let θ_1, θ_2, θ_3, ... be the angles which the same make with the horizon. Also let there be $2n$ beams, and let the length of each be a.

Then the mechanical equations for the n beams on the left of the vertical through the highest point D will be as follows:

$$\left.\begin{array}{l} Y_1 - Y_2 = W, \\ Y_1 a \cos\theta_1 + Y_2 a \cos\theta_1 = 2X a \sin\theta_1 \end{array}\right\} \text{.........} \ (1).$$

$$\left.\begin{array}{l} Y_2 - Y_3 = W, \\ Y_2 a \cos\theta_2 + Y_3 a \cos\theta_2 = 2X a \sin\theta_2 \end{array}\right\} \text{.........} \ (2).$$

$$\text{...}$$

$$\left.\begin{array}{l} Y_n = W, \\ Y_n a \cos\theta_n = 2X a \sin\theta_n \end{array}\right\} \text{.........................} \ (n).$$

It will be observed that there is no vertical pressure upon the upper end of the last or nth beam. Thus we have $2n$ equations involving $2n+1$ unknown quantities, viz. X, Y_1, Y_2, ... Y_n, θ_1, θ_2, ... θ_n. We must have *one* geometrical equation, which will result from the fact of the distance AA' between the points of support of the system being given; call it $2c$, then we have

$$a\cos\theta_1 + a\cos\theta_2 + \ldots + a\cos\theta_n = c \ \text{......} \ (n+1).$$

We can proceed a little further in the solution of the problem. We have

$$Y_n = W, \ Y_{n-1} - Y_n = W;$$

$$\therefore \ Y_{n-1} = 2W,$$

$$\text{in like manner } Y_{n-2} = 3W, \text{ and so on.}$$

$$\text{Lastly, } Y_1 = nW.$$

This result might have been foreseen, since the point of support A bears the whole weight of the beams, that is, the vertical pressure upon it is nW.

Again, we have

$$2X\tan\theta_1 = Y_1 + Y_2 = nW + \overline{n-1}\rceil W = \overline{2n-1}\rceil W,$$

$$2X\tan\theta_2 = Y_2 + Y_3 = \overline{n-1}\rceil W + \overline{n-2}\rceil W = \overline{2n-3}\rceil W,$$

$$2X\tan\theta_n = Y_n = W;$$

$$\therefore \frac{\tan\theta_1}{\tan\theta_2} = \frac{2n-1}{2n-3},\quad \frac{\tan\theta_2}{\tan\theta_3} = \frac{2n-3}{2n-5},\quad \&\text{c.},\quad \frac{\tan\theta_{n-1}}{\tan\theta_n} = 3.$$

We cannot carry the solution of the problem any further in its general form, because of the unmanageable character of the equation $(n+1)$.

XLVIII. Three uniform beams rest upon abutments, as in the preceding problem. Write down the equations necessary for the solution of the problem.

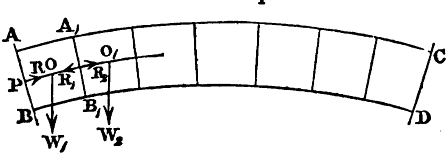
XLIX. Let AB, BC, CD be the three beams; and let $AB = BC$, and $AD = CD = 2AB$; then if a circle be described through B and C and touching the horizontal line through B, it will pass through the point of intersection of AB, DC produced.

L. Two equal beams rest upon a smooth horizontal plane, bearing upon each other at their highest points and having their lower extremities connected by a string of given length; find the tension of the string.

In common roofs the lower extremities of the principal rafters are frequently connected by a beam called a *tie-beam;* the result of the preceding problem will point out the use of the tie-beam, in preventing any spreading of the walls in consequence of the outward pressure caused by the weight of the roof.

LI. Supposing the surfaces of the stones or *voussoirs* of an arch to be perfectly smooth, to determine the conditions of equilibrium.

Let $ABCD$ be the arch, composed of perfectly smooth stones which press against each other, and rest upon solid masonry at AB and CD.



Consider the equilibrium of the first stone or voussoir $AB\,B_1A_1$, the weight of which call W_1, which acts through the centre of gravity. Now the pressures of the voussoir upon the stonework at AB will be equivalent to some one pressure R acting at a point P perpendicu-

larly to AB. Let the direction of this pressure meet that of W_1 in O; then the pressure between the first and second voussoirs must be a pressure R_1 passing through O: therefore if we draw OP_1 perpendicular to A_1B_1, this will be the direction of R_1. In like manner we may determine the direction of the action between the second and third voussoirs, and so on.

Let $\theta\theta_1\theta_2\ldots$ be the angles which AB, A_1B_1, A_2B_2, ... make with the vertical. Then we shall have the following equations,

$$\left.\begin{aligned} R\cos\theta - R_1\cos\theta_1 &= 0, \\ R\sin\theta - R_1\sin\theta_1 &= W_1 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

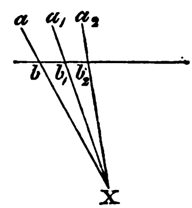
$$\left.\begin{aligned} R_1\cos\theta_1 - R_2\cos\theta_2 &= 0, \\ R_1\sin\theta_1 - R_2\sin\theta_2 &= W_2 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

&c. &c.

The analogy of these equations to those of the funicular polygon will at once be noticed.

We shall not pursue this investigation further, but will point out a simple geometrical construction by means of which we can determine, if the directions of the joints of the voussoirs be given, the relations of the weights, and *vice versâ*.

(1). Let the directions AB, A_1B_1, &c. be given.

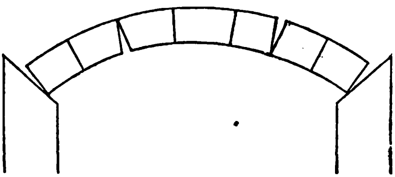


Through any point X draw Xa, Xa_1, Xa_2 ... parallel to AB, A_1B_1, A_2B_2, ... and draw any horizontal line cutting the above lines in $bb_1b_2\ldots$ respectively. Then the straight lines bb_1, b_1b_2, ... will be proportional in length to the weights $W_1W_2\ldots$ The student will easily supply the proof.

(2) Let the weights W_1, W_2, W_3, ... be given. The direction of the face of the abutment AB is arbitrary, let it be abX; and through any point X in it, draw a straight line parallel to the face of the other abutment, (not represented in the figure). Draw a horizontal line cutting these two, and divide the intercepted part in b_1, b_2, ... so that bb_1, b_1b_2, ... shall be in the same proportion as W_1, W_2, ... Join Xb_1, Xb_2, ...; these will be the required directions of the joints.

The preceding investigation, though useful as an illustration of statical principles, is of no practical value in the construction of arches. For the hypothesis of the smoothness of the voussoirs is one which

does not hold even approximately; the force of friction is necessarily great, and may be made as large as we please. An arch constructed upon the preceding principles would stand; but with this peculiarity, that any additional pressure to any one of the voussoirs would cause that voussoir to sink and others to rise, until the *line of pressure* had adjusted itself to the altered circumstances. But when the friction is such as to prevent the voussoirs from sliding upon each other, the arch cannot give way except by *breaking:* and a little consideration will shew that it must break into at least *three* portions, and the fractures must be alternately in the *extrados* and *intrados*, or external and internal curve of the arch. The true theory of the arch therefore is much more nearly connected with that of the equilibrium of beams, than that of the theoretical problem of an arch built with smooth voussoirs.



LII. A string having its extremities fixed to the ends of an uniform rod, of weight W, passes over four tacks, so as to form a regular hexagon; the rod, which is horizontal, being one of the sides; find the tension of the string and the vertical pressure on each tack. Shew also that the rod cannot hang in any other than a horizontal position.

LIII. If one of the highest tacks be removed, compare the vertical pressure upon the other highest tack with its value as obtained in the preceding problem.

LIV. Still further; of the three remaining tacks remove that which is furthest from the highest, and again calculate the vertical pressure upon the highest tack.

LV. A uniform beam rests with one end upon a given smooth inclined plane, and is supported upon a prop which is at a given distance from the inclined plane; determine the limits between which the length of the beam must lie in order that equilibrium may be possible.

LVI. If in the preceding problem the beam be too long for equilibrium to be possible, and instead of merely resting upon the inclined plane it be therefore attached at a given point by means of a hinge, find the direction and magnitude of the strain upon the hinge.

LVII. A ladder rests in a given position against a smooth wall with its foot upon a rough pavement; determine the weight which must be placed at its foot to prevent it from sliding; the coefficient of friction for the ladder and the weight being both given.

LVIII. An equilateral triangle of given weight is supported in a horizontal position by three equal threads; the strength of the threads is such, that two of them would just support the triangle; determine the greatest weight which can be placed upon the triangle without breaking the threads.

LIX. A ladder rests against a wall; given the weight of the ladder, the angle which it makes with the horizon, and the coefficient of friction both for the ground and also for the wall; find how high a man of given weight can ascend without causing the ladder to slip.

LX. A uniform string hangs in equilibrium over two pegs (not in the same horizontal line), shew by general reasoning that the two extremities must be in the same horizontal line.

First suppose that we have four pegs A, B, C, D, in the same vertical plane, of which A is vertically over C, and B vertically over D; and let C and D be in the same horizontal line, but A and B not in the same horizontal line. Take an endless string, and suspend it upon these four pegs; then it will arrange itself in some position of equilibrium, forming a festoon or catenary from A to B, and another from C to D; the catenary CD will be perfectly symmetrical with respect to C and D, and therefore will produce precisely the same tension at C as at D: whatever horizontal tension there may be at C and D will be counteracted by the pegs, and the vertical tension will be equivalent to two equal weights hung upon the strings AC, and BD: these equal weights may be represented by two equal pieces of string CE, DF; hence equilibrium will still subsist if we remove the pegs C and D, and the catenary CD, and instead thereof add the equal lengths of string CE and DF. We have now the string $EABF$ in equilibrium upon the two pegs A, B, with its extremities in the same horizontal line; and conversely it is not difficult to shew, that the string cannot be in equilibrium unless its extremities are in the same horizontal line.

Suppose A and B to be in the same horizontal line, then it is easy to calculate the length of AE and BF in order that there may be *equilibrium.* For we have already proved the general formula for the

tension $t^2 = s^2 + c^2$. Hence if $2s$ be the whole length of the string we must have $AE^2 = s^2 + c^2$.

LXI. A hemisphere is placed with its base in contact with a smooth wall, and it is moveable about an hinge at its lowest point; a string fixed to a point in the wall vertically over the hinge carries a weight, which presses the string against the hemisphere, and so preserves equilibrium: find the smallest weight which will answer the purpose. The distance of the point of attachment of the string from the hinge may be taken to be three times the radius.

When the system is in equilibrium, the student will observe that the string may be supposed to be fastened to the hemisphere at the two extreme points of the arc of contact, and therefore the action of the string will be reduced to that of two forces at those two points, each equal to the suspended weight.

LXII. A sphere rests upon two inclined planes; find the pressure sustained by each.

LXIII. An isosceles right-angled triangle rests in a vertical plane with the right angle downwards, between two pegs at a distance a from each other in a horizontal line; determine the positions of equilibrium.

LXIV. If G be the centre of gravity of a triangle ABC, three forces in the direction of and proportional to GA, GB, GC, will keep a particle at G at rest.

LXV. A hemisphere is supported by friction with its curved surface resting upon a horizontal and in contact with a vertical plane; find the limiting position of equilibrium.

LXVI. AB is a rod capable of turning freely about its extremity A, which is fixed, CD is another rod equal to $2AB$, and attached at its middle point to the extremity B of the former, so as to turn freely about this point; a given force acts at C in the direction CA, find the force which must be applied at D in order to produce equilibrium.

LXVII. If a set of forces, acting at the angular points of a plane polygon be represented in magnitude and direction by the sides, taken in order, shew that the tendency to turn the body about an axis perpendicular to the plane of the polygon is the same through whatever point of the plane the axis passes.

LXVIII. A triangular board of given weight rests in equilibrium with its base on a horizontal plane sufficiently rough to prevent all

sliding. A force acts upon it in its own plane and in a given line through the vertex and without the triangle; find the limits between which the magnitude of the force must lie if the equilibrium is preserved.

LXIX. Two equal circular disks with smooth edges, placed on their flat sides in the corner between two smooth vertical planes inclined at a given angle, touch each other in the line bisecting the angle. Find the radius of the least disk which may be pressed between them without causing them to separate.

LXX. If two scales, one containing a weight P and the other a weight Q, be suspended by a string over a rough sphere, and if Q be on the point of descending, then the weight $\frac{Q^2-P^2}{P}$ put into the opposite scale will make that scale to be on the point of descending.

LXXI. A uniform and straight plank rests with its middle point upon a rough horizontal cylinder which is fixed, their directions being perpendicular to each other. Find the greatest weight that can be put upon one end of the plank without causing it to slide from the cylinder.

LXXII. One end of a beam, whose weight is W, is placed on a smooth horizontal plane; the other end, to which a string is fastened, rests upon another smooth plane, inclined to the horizon at an angle (α); the string passing over a pully at the top of the inclined plane hangs vertically, supporting a weight (P). Shew that the beam will rest in any position if a certain relation hold between P, W, and α.

LXXIII. A cylinder, the base of which is in contact with a smooth vertical plane, is supported by a string fastened to it at a point of its curved surface whose distance from the vertical plane is b. Shew that b must be intermediate in value to $b - 2a\tan\theta$, and b, where $2b$ is the altitude of the cylinder, a the radius of the base, and θ the angle which the string makes with the vertical.

LXXIV. A flat board in the form of a square is supported upon two smooth props with its plane vertical; investigate an equation for determining its positions of equilibrium, the distance between the props being equal to half a side of the square.

The equation required is $\cos\theta - \sin\theta = \cos(2\theta + \alpha)$, where α is the angle which the straight line joining the props makes with the horizon, θ *that which one side of the square makes with the same.*

LXXV. A pack of cards is laid on a table; each projects in the direction of the length of the pack beyond the one below it; if each project as far as possible, prove that the distances between the extremities of the successive cards will form an harmonic progression.

LXXVI. A uniform slender rod passes over the fixed point A and under the fixed point B, and is kept at rest by the friction at the points A and B; determine the limiting position of equilibrium.

LXXVII. Four uniform slender rods AB, BC, CD, DA, rigidly connected, form the sides of a quadrilateral figure, such that the angle A is a right angle, and the points B, C, D are equidistant from each other; when the whole is suspended at the angle A, determine the position of equilibrium.

LXXVIII. A uniform bent lever, the arms of which are at right angles to each other, is just capable of being inclosed in the interior of a smooth spherical surface; determine the position of equilibrium.

It will be seen that the plane of the lever must be vertical, so that the directions of the forces will all lie in one plane.

LXXIX. Two unequal weights, connected by a straight rod without weight, are suspended by a thread of given length, fastened at the extremities of the rod, and passing over a fixed point; determine the position of equilibrium.

LXXX. A smooth body in the form of a sphere is divided into hemispheres and placed with the plane of division vertical upon a smooth horizontal plane; a string loaded at its extremities with two equal weights hangs upon the sphere, passing over its highest point and cutting the plane of division at right angles; find the least weight which will preserve equilibrium.

ELEMENTARY MECHANICS

DESIGNED CHIEFLY FOR THE USE OF SCHOOLS.

PART II.

DYNAMICS.

BY

HARVEY GOODWIN, M.A.,
LATE FELLOW AND MATHEMATICAL LECTURER
OF GONVILLE AND CAIUS COLLEGE.

CAMBRIDGE: JOHN DEIGHTON.
LONDON: SIMPKIN, MARSHALL & CO., AND GEORGE BELL.
LIVERPOOL: DEIGHTON & LAUGHTON.
M.DCCC.LIII.

Cambridge:
Printed at the University Press.

PREFACE.

THE following elementary treatise on Dynamics is the companion to the treatise on Statics published in 1851. It comprises those portions of the subject which can be conveniently treated without the direct aid of the Differential Calculus, and is composed upon the same method as that adopted for Statics.

I have made one deviation from the plan usually adopted in treatises on the principles of Dynamics, which I trust may be found to simplify the subject to beginners.

Instead of explaining and discussing the three laws of motion in one Chapter and then following them into their applications, I have taken each law in conjunction with those applications which are possible without anticipating any law which is to follow. Thus I have proceeded at once from the first law of motion to the theory of falling bodies, and to the second law of motion I have immediately appended the parabolical theory of projectiles. I imagine that this method of treatment will be found to alleviate the difficulties which are frequently found to belong to the introduction of the third law of motion.

In the preface to my elementary treatise on Statics I announced the intention of shortly publishing a brief treatise on the Conic Sections; my promise has not been fulfilled in consequence of unavoidable circumstances. I have referred in the present work to "Conics" in general, whenever I have had occasion to quote any of the properties of the Conic Sections: any treatise on the subject will supply all the propositions required.

H. GOODWIN.

CAMBRIDGE,
October, 1853.

TABLE OF CONTENTS.

ERRATA.

p. 40, line 7 *for* v *read* s.
... — ... 8 *for* v *read* v^2.
... 72 ... 6 *for* 1000 *read* 500.
... — ... 7 *for* 25000 *read* 12500.
... — ... 8 *for* 149 *read* 24.
... 106, Ex. 16, *for* shew that......motion *read* determine the motion of the centre of gravity.
... 126 line 9 *for* $v^2 + V^2$ *read* $v^2 - V^2$.

The following is a list of corrections to be made in the ELEMENTARY STATICS.

p. 23, line 10 from foot of page, *for* 14° *read* 4°.
... 25, line 4 from foot of page, for b *will be*, read h *will be*.
... 46, Prob. 16. In this problem it is sufficient to know the *sum* of the two weights P and Q.
... 92, Prob. 18, *for* 12 *read* 120.
... 164, line 11, *for* μR *read* R.
... 171. In equation (3) *for* P *read* p.
... 179. In Prob. 11, *for* $\cos(\alpha + \alpha')$ *read* $\operatorname{cosec}(\alpha + \alpha')$.
... 180. In Prob. 16 the length of the beam has been for simplicity's sake taken equal to the diameter of the sphere; in line 9, *for* $r =$ the radius of the sphere, *read* $a =$ the radius of the sphere, or half the length of the beam.
... 181, line 8, *for* $\frac{W'}{W}$ *read* $\frac{W'}{a^2 W}$.

INTRODUCTORY CONVERSATION.

Tutor. WHEN we commenced the study of Mechanics we divided the subject into two parts, *Statics* and *Dynamics;* I now propose to introduce you to the second of these divisions.

Pupil. *Statics* is that part of Mechanics which treats of *bodies at rest; Dynamics* that which treats of *bodies in motion.*

T. That is the proper distinction; force, you will remember, we defined to be any cause which produces, or tends to produce, motion in a body; if the motion is actually produced, the action of the force becomes a dynamical problem. If given forces act upon a body, the body will move in a certain definite manner, depending upon the nature of the forces, and our purpose is to determine that motion; or sometimes the motion is given, and the problem is to determine the nature of the forces under the influence of which the motion takes place. Before, however, we concern ourselves with these general problems, it will be necessary for us to consider the manner in which we can measure the motions of bodies; we speak, for instance, of a body moving in a *given* manner; how can we suppose the motion to be *given?*

P. We must know, I suppose, in what direction the body is moving, and how fast.

T. That is quite true; and to make the matter more easy to be understood, let us separate the two things, the direction in which the body is moving, and the rate at which it is moving; the direction may be changing from one instant to another, as the course of a ship is continually altering, but let us suppose the simple case of a

body moving in one direction, that is, in a straight line, from A to B for example. Suppose the distance from A to B to be one mile, and suppose that a body leaving A at 12 o'clock reaches B at 1 o'clock, then we should say that the body moved at the rate of 1 mile per hour. But it would not follow of necessity that at half-past twelve o'clock the body was exactly half-way between A and B; would it?

P. Certainly not; for the body might lag behind during the first half-hour, and might make up the lost time by increased speed in the second.

T. Manifestly so; in fact, if we made observation of the number of feet passed over by the body in each successive minute, we might perhaps not find the numbers for any two minutes to be the same. We should express this mathematically by saying that the *velocity* of the body was *not uniform*.

P. By velocity you mean the rate of the body's motion.

T. Yes; it is difficult to express the term velocity by any word more simple than itself: if one body move from A to B in an hour, and another in two hours, then I should say that the velocity of the one body was twice as great as that of the other, or that one moved twice as fast as the other. In saying this, however, I should assume that the body moved uniformly, or passed over the same number of feet in successive minutes and seconds; and this uniform motion in a straight line from one point to another is the simplest case of motion possible, but properly speaking there is nothing dynamical in such motion as this.

P. Nothing dynamical?

T. No; we spoke of Dynamics as the science of *force producing motion:* now when a body is moving uniformly and in a straight line we shall presently find that

no force is acting upon it, and therefore its motion has nothing to do with Dynamics.

P. What put it in motion then?

T. The question for us is not, what originally put it in motion, but what force is acting upon it *now;* and inasmuch as no force is acting upon it during its actual course from A to B, its motion between those points involves no considerations of the nature and action of force. The science of the motion of bodies, which does not introduce any considerations of force, is sometimes distinguished as the science of *kinematics.*

I said then, that the simplest case of motion was that of a body moving uniformly in a straight line; the case next in simplicity is that of a body moving in a straight line but not uniformly; in that case, if the nature of the force be given, the problem of Dynamics is to determine the rate of the body's motion at each instant, and its position at each instant, or if these be given, to find the nature of the force. Ascending in the scale of generality we have the case of a body moving neither uniformly nor in a straight line, and then similar problems present themselves, but involving greater difficulties. These problems will not be wholly solved in the treatise which you are about to read; the solution would require more mathematical resource than we can command; we shall be able, however, to solve many problems of great interest, and to exhibit clearly the principles of the subject. Other problems will present themselves, such as the motion of pendulums, and the collision of bodies impinging one upon another, which we can more or less completely solve. At present, however, let us confine our attention to the mode of estimating the velocity of bodies; this subject, as you will perceive, is treated by itself in the first chapter of the treatise, and the arrangement is adopted in order that you may remember that the question of the mode of measuring velocity is rather the introduction to Dynamics, than Dynamics itself.

CHAPTER I.

ON VELOCITY.

1. When a particle occupies the same position in space during successive instants of time the particle is said to be at rest, if otherwise the particle is in motion.

The motion of a particle may be either in a straight line or not, in other words it may be either rectilinear or curvilinear. For simplicity's sake let us first consider rectilinear motion.

2. Def. *If a particle pass over equal spaces in equal times its velocity is said to be uniform; if not, it is said to be variable.*

3. Uniform velocity is measured by the space passed over by the particle in any given time as an hour, a minute, or a second. The last-mentioned is the portion of time usually fixed upon, and is called the unit of time. To define a second accurately it would be necessary to have recourse to astronomical explanations, but to do so would not add to the clearness of our definition; it will be sufficient for us to take the expression *second of time* in its ordinary popular signification, as being the three hundred and sixtieth part of an hour as shewn by a good clock. A second so defined is taken as the unit of time, that is to say, other portions of time are measured by the number of seconds they contain; for example, 60 would represent 60 seconds or one minute, and generally the symbol t if it stand for time would represent t seconds or t units of time.

In like manner a *foot* is chosen as the unit of length, or of linear measure; the foot does not admit of accurate astronomical definition, it is an arbitrary length, determined by a certain standard, and fixed by Act of Parlia-

ment. It will be sufficient for us to refer to the ordinary notion of a foot as measured by a common measuring line, or carpenter's rule. Any number denoting length in the following treatise will, unless the contrary be expressed, denote so many *feet;* thus 6 would mean 6 feet, and generally s would stand for s feet or s units of linear measure.

4. Having thus defined our measures of space and time we can very simply define our measure of uniform velocity; for we say that uniform velocity is measured by the number of feet which a particle passes over in one second, or more generally, by the space described in a unit of time. For example, suppose a body moves over the space of 30 yards in 2 minutes; then it moves over 3 feet in 4 seconds, or three fourths of a foot in one second; consequently we should speak of its velocity as $\frac{3}{4}$. More generally, suppose that a body passes uniformly over a space s in the time t, that is, s feet in t seconds, then it passes over $\frac{s}{t}$ in one second or the velocity is measured by $\frac{s}{t}$.

5. It is usual to denote the velocity of a particle by the letter v. From what precedes, it immediately follows that if v be the velocity with which a body passes uniformly over a space s in time t, then $s=vt$. This also appears as follows; v by definition is the space described in one second, therefore vt is the space described in t seconds; but s is the space described in t seconds; therefore $s=vt$.

Hence of the three quantities s, v, t, any two being given the third may be found. For example: a body moves during 3 seconds with a velocity of 6 feet per second, how far will it move? Answer, 18 feet. And so in other examples which are too simple to render it necessary to adduce any more.

6. Next let us consider how velocity is to be measured when *not uniform,* that is, when the body does not describe

equal spaces in equal times. In this case we cannot measure the velocity by the space described in a second, or a unit of time, because the space described in different seconds will be different. Suppose, for instance, that we let a weight fall from any height to the earth, then it will be found to move more and more rapidly as it approaches the earth, and therefore its velocity is not uniform; if we were able to observe its motion accurately, we should find that in the first second of its fall it moved through a little more than 16 feet, in the next second we should find that it moved through more than 48 feet, in the third through more than 80, and so on; we could not, therefore, speak of the space described in one second, because the spaces described in successive seconds rapidly increase. How then shall we measure the velocity at any instant, at the end of the first second for example? It is clear that the proper measure of the velocity at that instant is the space which the body *would* describe in a given time, if it continued to move with the velocity in question; now the velocity of the body at the end of the first second is found (it matters not at present how) to be such that a body proceeding with that velocity would in a second pass over 32.2 feet, consequently we say, that the body has a velocity of 32.2 feet per second. At the end of two seconds it is in like manner found to be moving at such a rate, that if it moved for a second with that velocity it would pass over 64.4 feet, consequently at the end of the second second it is said to have a velocity of 64.4; and so on. This method of measuring velocity will not occasion any difficulty, if the student considers that it is in fact the method by which even the most uninstructed person would estimate velocity; suppose, for instance, that a person is in a railway carriage, he guesses the velocity of the train to be 30 miles per hour; what does he mean by speaking of a velocity of 30 miles per hour? he intends to express that the train is moving at such a rate that *if* it moved uniformly at the same rate during one hour it *would* pass over 30 miles.

7. Hence we can give the following formal statement

of the manner in which velocity is measured. *The velocity of a body, if uniform, is measured by the space which the body passes over in an unit of time; if variable, the velocity at any instant is measured by the space which would be passed over in a unit of time, if the body moved during that unit of time with the velocity which it has at the instant in question.*

8. If the velocity at the end of successive units of time is increased or diminished by equal quantities, then the velocity is said to be *uniformly accelerated* or *retarded* respectively. For example, when a weight is allowed to fall to the earth the velocities at the end of the 1st, 2nd, 3rd, ... seconds are found to be 32.2, 2×32.2, 3×32.2, ...; or the velocity is increased in successive seconds by the same quantity, 32.2 feet; consequently the motion of a falling body is said to be a case of *uniformly accelerated motion*. If a body were projected upwards from the earth's surface, we should have a case of uniformly retarded motion during the ascent of the body, and of uniformly accelerated motion during the descent. This kind of motion is next in order of simplicity to that of uniform velocity.

9. The motion of a body at any instant is entirely determined, if we know the *direction* in which the body is moving, and the *velocity* with which it is moving. For instance, if we know that a body is moving in a direct line from one point A towards another B, with a velocity of 10 feet per second, the motion of the body will be completely determined. And this will be the case even though the motion of the body be not rectilinear; for if a body is moving in a curve, the direction of the body's motion at any point is the direction of the tangent of the curve at that point, and if the direction of the tangent and the magnitude of the velocity be given, the motion will be fully determined. We will now explain another method, slightly different from this, by which the motion of a body may be determined, and which will be found hereafter of very great utility.

10. Suppose, for simplicity's sake, that a body is moving uniformly down an inclined plane; then if the inclination of the plane be given, and the velocity of the body, we shall know the whole motion of the body, and at any moment it will be possible to assign the body's precise place. Again, suppose that through the upper extremity of the inclined plane we draw a vertical and a horizontal line; then if we know the position of the body at any moment we shall know its distance from each of these lines, and conversely, if we know its distance from each of these lines we shall know its position; and if we know its velocity down the inclined plane, we shall know the rate at which its distance from each of the lines spoken of increases, and conversely, if we know the rate at which these distances increase, we shall know the velocity and also the inclination of the plane.

Let us make this more clear by a numerical example; suppose the plane to be inclined at an angle of 45° to the horizon, and the body to be descending with a velocity of 1 foot per second; then at the end of the first second the body will have descended through 1 foot along the inclined plane, and its distance from each of the lines will be $\frac{1}{\sqrt{2}}$ feet; in like manner at the end of 2 seconds the body will have moved through 2 feet along the inclined plane, and its distance from each of the lines will be $\sqrt{2}$ feet; and so on. Now conversely, suppose that we know that the distances of the body from the two lines are always equal to each other, and increasing uniformly at the rate of $\frac{1}{\sqrt{2}}$ feet per second, then we may conclude that the body is in fact moving upon a straight line inclined to the horizon at 45° with a uniform velocity of 1 foot per second.

It would seem to be an obvious mode of describing briefly what precedes, to say that the body in question has *a uniform horizontal velocity* of $\frac{1}{\sqrt{2}}$ feet per second, and a

uniform *vertical velocity* of the same amount; and it will come to the same thing whether we speak of a body having a uniform velocity of 1 foot per second down a plane inclined at an angle of 45^0 to the horizon, or whether we speak of the body having a horizontal and vertical velocity, each equal to $\frac{1}{\sqrt{2}}$ feet per second. If, therefore, we speak of a body having two velocities in two assigned directions, the student will, it is hoped, find no cause of confusion in the expression; the expression is perfectly free from any source of confusion, if only he distinctly bear in mind the sense attached to the terms.

Now let us generalize the method which we have illustrated in the preceding example. Suppose a particle to be moving with a velocity v down a plane inclined at an angle θ to the horizon, and suppose a horizontal and vertical line to be drawn through the highest point of the plane as before. Then the distances of the particle from the highest point of the plane at the end of the 1st, 2nd, 3rd, ... seconds will be v, $2v$, $3v$,... respectively; and the distances of the particle from the horizontal and vertical lines will be at the same instant $v \sin\theta$, $2v\sin\theta$, $3v\sin\theta$,... and $v\cos\theta$, $2v\cos\theta$, $3v\cos\theta$,... respectively. Hence it will be perfectly intelligible, and will perfectly describe the body's motion, if we say that it is moving with a *vertical velocity* of $v\sin\theta$, and a *horizontal velocity* of $v\cos\theta$.

Conversely, if we speak of a body having a vertical velocity V and horizontal velocity V', the motion of the body will be entirely determined; that is, the direction and magnitude of the velocity will be known. For let v be the actual velocity, and θ the angle which the direction of the body's motion makes with the horizon, then we shall have from what precedes,

$$v\sin\theta = V, \qquad v\cos\theta = V',$$

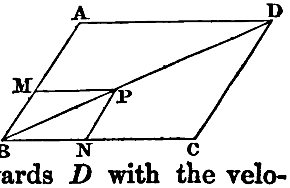
$$\therefore \tan\theta = \frac{V}{V'}, \qquad \text{and } v^2 = V^2 + V'^2.$$

that is, the direction and magnitude of the velocity are both determined.

11. This subject may be exhibited from a rather different point of view, and may be further illustrated by introducing a geometrical method of representing velocity, analogous to the method used in Statics for representing force. As in statics we made use of a finite straight line given in position to represent a force of which the magnitude and direction were known, so we may in like manner make use of a straight line to represent a velocity; for velocity though a very different thing from force is nevertheless like force in this, that it is entirely determined when its magnitude and its direction are known, and these two elements can both be represented by a straight line given in magnitude and in position, and consequently such a straight line may be used as the geometrical symbol of velocity.

And the student who understands what has been already said concerning the velocity of a body being capable of being represented by two velocities in definite directions, and who compares the formulæ of the preceding article with formulæ for the composition and resolution of forces*, will not be surprised to find that velocities admit of composition and resolution in a manner precisely similar to forces, and that we have in fact a Parallelogram, a Triangle, and a Polygon of velocities.

Let $ABCD$ be a parallelogram, of which BD the diagonal represents a certain velocity v in direction and magnitude. Then what we mean by saying that BD represents v in direction and magnitude is this,



that a particle moving from B towards D with the velocity v would reach D in one second. Now take any position P of the particle between B and D and draw PM, PN parallel to BC and BA respectively; then if we call the rate at which the line PM increases the *velocity*

* *Statics*, Chap. I. Art. 14, and Chap. IV. Art. 8.

parallel to BC, and the rate at which PN increases the *velocity parallel to AB*, the term *velocity parallel to a given line* will be quite definite; and since BD is the space actually described in one second and is therefore the geometrical representation of the velocity of the particle, AD and CD, or BC and BA which are respectively equal and parallel to them, being the distances described by the particle parallel to the directions of the sides of the parallelogram may be spoken of as the velocities parallel to those sides.

Conversely, if we conceive of a particle as having simultaneously two velocities, and if those velocities be represented respectively in magnitude and direction by two straight lines, then the actual resultant velocity will be represented by the diagonal of the parallelogram described upon these two straight lines as sides.

12. We may give this proposition a trigonometrical form.

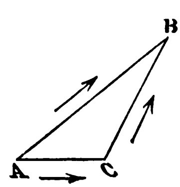
Let V, V', be the velocities of a particle parallel to two straight lines inclined at an angle a, and let v be the resultant velocity; in fact in the figure of Art. 11, let $AB = V$, $BC = V'$, $ABC = a$, $BD = v$; then

$$\text{since} \quad BD^2 = AB^2 + BC^2 + 2AB \, . \, BC \cos ABC,$$

$$\text{we have} \quad v^2 = V^2 + V'^2 + 2VV' \cos a.$$

13. We might call V and V' the *resolved parts* of v in the directions BA, BC respectively; we usually however reserve the expression *resolved part* for the case in which the directions are *at right angles* to each other, using in other cases the term *component* velocity.

14. The *triangle* of velocities may be enunciated as follows, and requires no demonstration as it is in fact the *parallelogram* in another form. If two sides of a triangle AC, CB represent in magnitude and direction the component velocities of a particle in two given directions, then the third side AB



represents the actual velocity of the particle both in magnitude and direction.

15. Also if we suppose three velocities represented in magnitude and direction by the three sides of a triangle taken in order, AC, CB, BA, (*not* AB) to be simultaneously impressed upon a particle at rest, then the particle will remain at rest.

16. In like manner, if any number of velocities be supposed to be impressed simultaneously upon a particle at rest, and if these velocities be represented in magnitude and direction by the sides of a *polygon* taken in order, then the particle will remain at rest.

17. It is to be observed, that in what has been here said, the student is not to consider the causes to which the velocities spoken of are due; that is, he is not to consider the *forces* necessary to produce the velocities, nor according to what laws such forces act; these considerations belong to a subsequent part of the subject; when we speak of two velocities being impressed upon a body, we merely intend to express, that the body is so made to move that its velocities in the assigned directions shall be such as are required. And in like manner, when we speak of any number of velocities being impressed, we merely mean that the velocities in the assigned directions shall be as described.

Thus the student will perceive, and it is highly desirable that he should bear in mind, that hitherto we have been concerned with a part of the subject which is *purely geometrical*, and this chapter on the composition and resolution of velocities, though a necessary introduction to Dynamics, is, properly speaking, not itself *dynamical.*

CONVERSATION UPON THE PRECEDING CHAPTER.

P. I am not sure that I completely understand what is meant by the statement that the subject of this chapter is purely geometrical; I find it difficult to conceive of a particle moving without thinking of some cause of its motion, and that cause must, I suppose, be a force.

T. A body may be in a state of uniform motion, as we shall see in the next chapter, without the action of any force; and the questions, how is velocity generated in a body at rest? and how is velocity changed in a body in motion? are the chief questions to be answered in Dynamics; but in the chapter you have just read there is no question, as to how velocity is generated or destroyed or changed, the purpose of the chapter is entirely to explain in what manner the velocity of a body is *measured.* And in considering the most convenient manner of doing this we are led to the method of *resolving velocities*, or of regarding the motion of a body as though it consisted of two motions parallel to two fixed directions; now this is a matter of mere convenience, and involves no mechanical principle whatever.

P. Yet we have a parallelogram of velocities, exactly as we had a parallelogram of forces.

T. That is the very reason why it is necessary to insist upon the purely geometrical nature of the parallelogram of velocities; in demonstrating the parallelogram of forces in Statics we were obliged to have recourse to certain axioms concerning force, now we have nothing of this kind in the case of the parallelogram of velocities; for velocity is not an agent like force, not a cause of motion, but a quality of the motion however produced; and the parallelogram of velocities merely states that the motion of a body may be estimated in two ways, which are equivalent to each other, and of which sometimes one will be found more convenient and sometimes the other.

P. It seems strange that there should be two propositions so similar, yet totally unconnected in principle.

T. I do not say that they are totally unconnected in principle, but to determine the exact relation of them would lead us into a rather difficult discussion; but I wish you to see, that whatever may be said of the parallelogram of *forces*, the parallelogram of *velocities* is certainly *not* mechanical, and is in fact a much simpler proposition than the parallelogram of forces, although in our arrangement of the subject of Mechanics this latter is treated first. If we met with the parallelogram of velocities as a proposition altogether unconnected with the parallelogram of forces, there would be no occasion to lay any stress upon its geometrical character; as it is there is considerable danger of beginners mistaking the nature of the proposition, and regarding velocity as something which acts upon a body in a manner analogous to force.

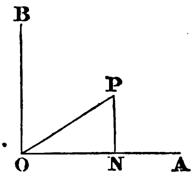
P. Does not velocity act upon a body?

T. No, the expression is absurd; yet I have frequently heard it used by beginners, and therefore am not sorry to have an opportunity of pointing out its erroneous character; it would be quite as correct to say that rest acted upon a body which does not change its position, as to say that velocity acts upon a body which does. There is a phrase frequently used in French works on Mechanics which seems to me singularly happy in expressing the right notion; it consists in speaking of a body as *animated* with a certain velocity; this expression clearly points out that the velocity is something which belongs to the body, not something which acts upon it from without. A body is *acted* upon by a *force*, it is *animated* with a *velocity*.

P. I believe that I understand what you wish to express, and that I am to regard it as immaterial whether the direction and magnitude of the velocity of a body be given, or whether it be supposed to be animated with two velocities in given directions simultaneously.

T. *Yes;* and you will perceive that these two methods

of considering the body's motion exactly correspond to the two following methods of determining the position of a body at rest. Let P be a point in the plane of the paper, the position of which we desire to describe. Take O any fixed known point, and draw two straight lines OA, OB in given directions; we will suppose them for simplicity's sake, to be at right angles to each other, though it is not necessary to do so. Join OP, and draw PN perpendicular to OA. Then the position of OA being known, that of P will be entirely determined if the distance OP be given and the angle POA; in fact, if we regard P as a spot in a field to which we wish to direct a person who is standing with us at A, and if OA be the hedge bounding the field, we have only to make the person start in a certain direction from O, and tell him to walk so many yards in that direction, and that he will then be at P. But the position of P will be also determined if we suppose ON and PN to be known, or the perpendicular distances of P from the two lines OA and OB; to adopt the preceding illustration, we could equally well guide a person from O to P by telling him to walk a certain distance ON alongside of the hedge OA, and then to turn to the left, and walk so many yards directly from the hedge. It is evident that these two modes of directing a person from O to P are, mathematically speaking, the same thing, and the two modes of measuring velocity correspond to the two modes of determining the position of P.

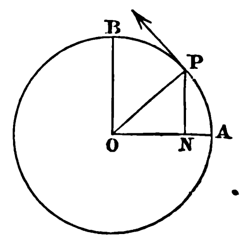


P. And why is one better than the other?

T. The great advantage of considering the motion of a body as determined by two velocities in given directions, instead of being determined by the actual velocity and the actual direction of its motion at a given time, is most apparent when we consider the case (which indeed is the most usual one) of a body moving, not in a straight line, but in a curve; when a body is moving in a straight line OP, it will generally be as convenient to give the angle which OP makes with a fixed direction OA and the velocity

along OP, as to give the component velocities parallel to OA and OB; but when the body is moving in a curve, you will notice that not only is its distance OP from the fixed point O continually changing, but also the angle POA is changing, and thus we have two quantities of quite different kinds, namely a straight line OP and an angle POA, the changes of which we have to consider; whereas by the other method we have the changes of two straight lines only to consider, and the curvilinear motion is reduced to two rectilinear motions.

By way of illustration let us suppose a body to be moving uniformly in a circle, and let us estimate its velocity in each way.



Let O be the centre of the circle; OA, OB two fixed straight lines at right angles to each other; P the position of the body at any given time; PN perpendicular to OA.

Suppose the motion of the body to be opposite to that of the hands of a clock, as indicated by the arrow in the figure, and let V be its velocity, and let $PON = \theta$.

Then in this case the distance OP of the body from O does not change, since P, by hypothesis, moves in a circle, and the only quantity which changes is the angle θ, which increases uniformly. Referring the motion to the two straight lines OA and OB we may speak of the body having the velocity $V \sin \theta$ parallel to OA, and $V \cos \theta$ parallel to OB; or rather we should speak of the velocity parallel to OA being $- V \sin \theta$, since it will be seen that as the body revolves the line NO decreases, and if, as is usual, we measure a velocity as positive when the distance from a given point increases, we must call it negative when that distance diminishes.

P. What do you mean by the angle θ increasing uniformly?

T. I mean that in each second, or each unit of time,

it increases by the same quantity. In fact, let the body begin to move from A, and at the end of one second let a be the angle which OA makes with the line joining O and the body, then if t be the number of seconds which have elapsed when the body arrives at P, we must have

$$\theta = at,$$

and this formula expresses mathematically that θ increases uniformly.

I may mention to you that the rate at which the line OP revolves, whether uniform or not, is called the *angular velocity* of P; in this case we should say that the *angular velocity* was *uniform*, and that a was the measure of it.

And I may further remark that if r be the radius of the circle, we have the relative

$$V = ra,$$

or, linear velocity = radius × angular velocity;

this at once follows from the trigonometrical equation, arc = radius × circular measure of angle; for the velocity is the rate of increase of the arc, and since the arc varies as the angle the rate of increase of the one varies as the rate of increase of the other.

And finally you will perceive from what I have said, that the position and motion of the body at the time t will be expressed indifferently in the two following ways;

(1) The body moves in a circle of radius r with a velocity V;

(2) The velocities of the body in the directions OA, OB, at the time t, are $-V\sin\frac{Vt}{r}$, $V\cos\frac{Vt}{r}$.

P. I think the term *linear velocity* is new to me.

T. It is merely used to distinguish velocity in the ordinary acceptation of the word from *angular* velocity; all velocity is linear velocity, or velocity measured by the rate at which a body moves, unless the contrary be stated.

What I have now said to you concerning the motion of a body in a circle has been merely with a view to illus-

tration; in consequence of the distance OP never changing, and the velocity entirely depending upon the change of the angle POA or θ, nothing can be simpler than to describe the motion in the first of the two ways which I have just shewn to you; but in many cases of motion, as you will see more clearly when you read Chapter III., a problem which would present great difficulties if we estimated the velocity by giving its actual magnitude and direction, becomes perfectly simple when its motion is considered as compounded of two velocities in fixed directions. In fact we shall find that the notion of resolution of velocity is as important in Dynamics as that of the resolution of force was in Statics.

P. Before we finish this conversation let me say that I think I should understand the reasoning of Art. 10 better if a figure had been supplied.

T. If a figure be necessary, supply one for yourself; nothing is easier, and you will find it always desirable as much as possible to construct your own figures according to the directions given, rather than content yourself with the figures which books supply. In the present instance take AB a straight line inclined at an angle θ to the horizontal line BC, so that $ABC = \theta$: through A the upper extremity of the plane draw AD horizontal, that is, parallel to BC; upon AB set off AP_1, P_1P_2, P_2P_3, ... each equal to v, then P_1, P_2, P_3, ... are the positions of the body at the end of the first, second, third, ... seconds respectively; draw P_1N_1, P_2N_2, P_3N_3, ... perpendicular to AD, then AN_1, AN_2, AN_3, ... are equal to $v\cos\theta$, $2v\cos\theta$, $3v\cos\theta$, ... respectively; and P_1N_1, P_2N_2, P_3N_3, ... are equal to $v\sin\theta$, $2v\sin\theta$, $3v\sin\theta$, ... respectively. Thus you will have a figure such as you require.

EXAMINATION UPON CHAPTER I.

1. Distinguish between uniform and variable velocity.
2. *How* is uniform velocity measured? how variable?

3. If s be the distance described in t seconds by a body which moves uniformly in a straight line with a velocity v, prove that $s = vt$.

4. If a be the distance of a body from a fixed point when it begins to move, and b its distance after it has been in motion n seconds with a velocity u, then

$$b = a + nu.$$

5. If one foot be the unit of length, and one second the unit of time, what will be the mathematical measure of the velocity of a body which passes over 8.76 miles in 1 hour, 3 minutes, 40 seconds?

6. With the same units what is the measure of the velocity of light, which requires 8 minutes to pass from the Sun to the Earth, taking that distance to be 95,000,000 miles?

7. The velocity of a falling body according to the ordinary conventions respecting the units of space and time is at the end of one second measured by 32.2; what would be the measure of its velocity if a mile and an hour were the units of space and time respectively?

8. And what must be the unit of space in order that the unit of velocity may be the velocity of a falling body at the end of one second, a second being the unit of time as usual?

9. Explain what is meant by compounding and resolving velocities.

10. A body moves down an inclined plane, and (owing to some cause unknown) it is found that the horizontal and vertical velocity increase uniformly with the time; given that the body has a velocity of one foot at the end of the first second, find the velocity at any other given time.

11. Is it possible to make a body move upon an inclined plane in such a manner that the horizontal velocity shall be constant, and the vertical velocity increase uniformly with the time?

12. A body falls down an inclined plane and it is observed that the vertical velocity varies as the time; shew that the actual velocity of the body must also vary as the time.

13. What is meant by a *negative* velocity? Give an illustration.

14. Enunciate the *parallelogram*, the *triangle*, and the *polygon* of velocities.

15. What is meant by *angular* velocity? If a ball at the end of a

string make a revolution in one second, what will be the measure of the angular velocity, taking as the unit of angle that angle which is subtended by an arc equal to radius?

16. What is the measure of the angular velocity of the hour hand of a clock? and if the length of hand be three inches, what is the velocity of the extremity?

17. Taking the radius of the earth to be 4000 miles, what is the velocity of a point at the equator?

18. Supposing the earth to move in a circle, radius 95,000,000 miles, about the sun, and the year to consist of 365 days 6 hours, find the earth's velocity.

19. A body descends uniformly down a plane, 1 mile long, inclined at an angle of 30° to the horizon, in 1 hour 4 minutes; find the vertical velocity.

20. Two railway trains meeting are observed to be 4 seconds in passing each other, had they been moving in the same direction they would have required 12 seconds; supposing the length of the trains to be 110 and 130 feet, find at what rate per hour each train is moving.

21. A body revolves uniformly in a circle, the plane of which is vertical; the time of revolution is 3 seconds, and the radius of the circle 6 inches; find the horizontal and vertical velocity for any given position of the body.

22. A body revolves uniformly in a circle, while the centre of the circle also moves uniformly in a straight line; shew how to express the velocity of the body at any instant.

23. A ball is thrown with a uniform velocity against a wall, and rebounds with only three-fourths of its velocity; supposing the distance from the point of projection to the wall to be 4 feet, and the time elapsing between the projection of the ball and its return to the point of projection to be .75 of a second, find the ball's velocity.

24. Compare the velocities of the extremities of the hands of a clock, the length of the hour hand being 2.2 inches, and that of the minute hand 3.6 inches.

CHAPTER II.

ON THE RECTILINEAR MOTION OF A SINGLE PARTICLE UNDER THE ACTION OF A SINGLE UNIFORM FORCE. FIRST LAW OF MOTION. FALLING BODIES.

---

1. ALL the matter in the universe is governed by fixed laws, and it is the great purpose of science to discover and develope those laws. The task has been one of great difficulty, and the progress which has been made hitherto has only been effected by means of the greatest diligence and perseverance on the part of men of consummate genius. Sir Isaac Newton was led by his investigations to propound the great law of universal gravitation, which is at present the greatest generalization of our knowledge which has been effected, and which, it may be added, has been continually confirmed since the discoverer's time. This law states that every particle of matter in the universe is acted upon by every other particle, according to a certain law which we shall not enter upon just now; thus a stone let fall descends to the earth because the earth attracts the stone, and in like manner the stone attracts the earth; the earth attracts the moon and the moon the earth: and so on. Hence the supposition of a single particle of matter unattracted by any other particle is an impossible supposition, there is no such thing in the universe as a single independent particle; but *if* there were, what would be the laws of its motion? To this question the following answer is given:

A particle, not acted upon by any external force, will either remain at rest or else move uniformly in a straight line.

2. We have already spoken of force as that which changes or tends to change the state of a body as regards rest or motion, and it might therefore seem unnecessary to

state that a particle not acted upon by a force will be at rest or else move uniformly in a straight line; but how do we know that no force resides in the particle itself? appearances would seem to favour such a notion; if we throw a ball under any circumstances whatever it soon stops, may not this be because there is some retarding force inherent in the ball? or again, if we throw a ball it always describes a curvilinear path; may not this be because there is some internal cause of deviation from a straight line? These objections may be removed by considering, that whenever there is a deviation from uniform and rectilinear motion, the deviation may be accounted for; thus friction and the resistance of the air generally account for bodies projected being brought to rest, and the attraction of the earth accounts for the deviation of a ball from a straight course; but we wish the student to see that the law which we have enunciated, and which is commonly called the *First Law of Motion*, is not so obvious as it may seem to him to be; and if he thinks that it is difficult to conceive of any inherent cause of change of motion residing in a particle, he may be asked whether it is not quite as difficult to conceive of any inherent power of changing the motions of other particles; yet this latter power is proved to exist; it may be shewn, beyond all doubt, that every particle acts upon every other particle however distant or under whatever circumstances, and with this fact before us we must be very cautious as to how we assert things to be conceivable or inconceivable. We do not however wish to do more than caution the student against rash conclusions, and to assure him that he has in this apparently simple law the result of the labours of centuries.

3. Take a smooth grass-plot and roll a ball upon it; if the ball be true in form, and the grass-plot even, you will have no difficulty in rolling it straight to any given point; if after having been started in the proper direction, *you observe it to* swerve, you at once conclude either that

the ball has a bias, or that there is an unevenness in the ground. The more nearly satisfied are the conditions of the perfect evenness and smoothness of the ground and the perfect regularity of the ball, the more nearly will the first law of motion be exemplified. It is by experiments such as this that we can most easily arrive at the truth of the law, and when the law itself has been once suggested, we shall seldom find difficulty in accounting for any apparent deviations from it in practice by reference to disturbing causes.

4. One of the most satisfactory proofs of the truth of this and of other laws arises from the comparisons with careful observations of results which are founded upon them and deduced from them by strict mathematical reasoning. The best illustration of the meaning of this remark is the case of the motion of the heavenly bodies; the motion of the moon and of the planets is calculated upon the hypothesis of the truth of the first law of motion and of other laws, and the motion so determined is found to agree in the most minute manner with the results of the most accurate observation; and modern observations are conducted with so much skill and furnished with instruments of such extreme delicacy that the smallest deviation from truth in any of the fundamental laws would certainly be sooner or later detected. On this ground, if on no other, we may in the present state of science feel sure of the accuracy of the law which has been above enunciated.

5. If then a body move uniformly and in a straight line we know that no force is acting upon it; if on the other hand it move in a straight line but not uniformly, we know that there *is* a force acting upon it. We must now consider how such a force is to be measured. For simplicity's sake we will at present confine our attention to the case of a body whose motion is uniformly accelerated, that is, which has equal additions of velocity in equal times; for instance, suppose a body to be moving at the end of one second with a velocity of 10 feet per second, at the end of two seconds

with a velocity of 12 feet, at the end of three seconds with a velocity of 14 feet, and so on, then a velocity of 2 feet is added in each successive second and the velocity is said to be uniformly accelerated by that amount. The rate at which the force adds velocity to the body's motion, in other words the amount of velocity which the force generates in one second, is evidently a proper measure of the intensity of the force, a proper measure at least *provided the magnitude of the body upon which the force acts be given.* In fact if we take two bodies in every respect similar and equal, and we find that a certain force P acting upon one of them adds a velocity V to its motion in each successive second, and that another force Q acting upon the other adds $2V$ to its motion in each successive second, then we may say that $Q = 2P$; or more generally, if P generates a velocity V in each second, and Q a velocity V', then

$$P : Q :: V : V'.$$

6. If we considered how P and Q should be compared when they acted upon two different bodies we should introduce a much more difficult question; this question we shall discuss afterwards; but at present, inasmuch as there is a large class of problems which can be solved without regard to the magnitude of the body in motion, it will be convenient to confine our attention to the mode of measuring force above explained, or which comes to the same thing, to confine our attention to those cases of motion in which we may regard the magnitude of the body as given, or in which the result is independent of the magnitude of the body. For instance, it appears from experiment that all bodies falling to the earth fall in such a manner that equal velocities are added in equal times, whatever be the magnitude of the bodies: this at least is true except so far as a deviation is introduced by the resistance of the air, a guinea and a feather do not fall to the ground with equal rapidity in the air, but they do so when the experiment is made under the exhausted receiver of an air-pump. The reason of this will be seen better hereafter, at present it

will be sufficient to refer to the fact, and to draw the conclusion that all problems of falling bodies may be treated without any other measure of force than that which has been above explained.

7. We have spoken of the effect of a force being different according to the *magnitude* of the body upon which it acts; technically we should have spoken of the *mass* of the body; the accurate definition and mode of estimating a body's *mass* we shall not consider at present, it will be sufficient for the student at this stage to understand by *mass* the *quantity of matter* which a body contains, without considering in what manner that quantity is to be accurately measured.

8. We should say then that the complete measure of a force involves the consideration of the mass or quantity of matter which it moves, and the rate at which it increases or diminishes the velocity of the said matter. Supposing the mass given, or supposing the problem to be such that the mass is indifferent, there remains to us only one effect of force, and that may be described properly as its *accelerating* effect, not intending to oppose that term to the term *retarding*, but considering it as inclusive of this latter term, and as describing what may be called the velocity-effect, whether velocity be generated or destroyed.

It is usual, however, to describe force in a manner rather different from that which has been given above, and which we have introduced thus cautiously for fear of conveying to the student an erroneous impression. It is usual to speak of force either as *accelerating force* or as *moving force*, that is, to speak of force either with reference merely to the acceleration it produces in a given body, or with reference to the acceleration as it would be modified by a change in the mass of the body upon which the force acts.

The formal distinction between *accelerating* and *moving* force may be given as follows:

Force considered only with reference to velocity generated, and not with reference to the mass moved, is termed accelerating force.

Force considered with reference to the mass moved, as well as the velocity generated, is termed moving force.

The impression against which we wish to guard the student is that of there being two different kinds of force, which is not true. Force is accelerating force, or moving force, as it is measured; every force is, in fact, a moving force, but if we confine our attention to a particular effect produced, then we may speak of the accelerating effect of the force, or more briefly of the *accelerating force*. This nomenclature is thoroughly recognized, and is very convenient, but requires to be well understood in order to prevent mistakes.

9. Using this nomenclature then, we say that we shall confine ourselves in the present chapter to the case of *accelerating force*, and we may give the following definition of the manner in which accelerating force is measured.

Accelerating force, if uniform, is measured by the velocity generated in a unit of time; if variable, by the velocity which would be generated in a unit of time if the force were continued constant during that unit.

This mode of measuring accelerating force, it will be observed, is quite parallel to that of measuring velocity; and the remarks which were made with respect to velocity will therefore assist the student in understanding the mode of measuring force. If the force be uniform there is no difficulty, because the velocity generated in a given time, as one second, is the effect of the force, and therefore the measure of it; but if the velocity generated be different for different seconds, we must adopt as the measure not the actual effect produced, but the effect which would be produced if the force were to continue throughout a unit of time the same as it is at the instant at which we desire to measure it. We shall be chiefly concerned with the

action of uniform force, at the same time it is desirable that the student should thoroughly comprehend the principle upon which variable force must be measured.

10. The most familiar instance of an uniform force is that of gravity, or the force which causes bodies to fall to the earth, to which reference has already been made. Let us apply our principle of measuring accelerating force to this case: suppose a body to fall from rest, it will be found by experiment that at the end of one second it has fallen through 16.1 feet as nearly as may be, and the velocity with which it is proceeding at this instant is the measure of the accelerating force of gravity. How is this velocity to be measured? it is not a constant velocity, and therefore we cannot measure it by the space which the body describes in the next second; but if we suppose that gravity ceases to act and that the body therefore proceeds after the first second with unchanged velocity, then the number of feet which the body *would* pass over in one second upon this hypothesis is the measure of gravity. We shall not concern ourselves just now with the question how this space is to be measured, but shall merely state that the number of feet which the body would pass over would be 32.2; hereafter we shall explain how this quantity is accurately determined. We have then 32.2 feet as the measure of the accelerating force of gravity, it being understood that the unit of time is one second; for convenience' sake the quantity 32.2 is usually denoted by the letter g, and in all which follows therefore, unless the contrary be stated, the student will remember that g represents the accelerating force of gravity, and that it may always be replaced by the number 32.2.

11. We shall now consider this problem; how far will a body proceed in a given time under the action of a given accelerating force? or, to put the same thing in a more familiar form, a body falls to the ground, through how many feet will it fall in a given number of seconds? It may be said that this is matter for experiment, and

that we can know nothing concerning gravity without making observations upon falling bodies: what we are about to prove however is this, that there is a certain simple relation existing between the measure of the accelerating force and the space through which a body falls in a given time; if this space be observed the accelerating force will be known; if on the other hand the accelerating force has been determined by any other means, then the space described in a given time will be known without any further experiment. For instance, we stated that under the influence of gravity a body will fall through 16.1 in one second, and the velocity which it then has, in other words the accelerating force, is 32.2; it cannot have escaped the student's observation, that one of these quantities is double of the other; now it will appear from the investigation which we are about to give that this is not a fortuitous relation, but that in general, accelerating force when uniform is measured by twice the space described by a body under the action of the force in one second.

12. Prop. *If* s *be the number of feet through which a body moves from rest under the action of an accelerating force* f *in* t *seconds, then* $s = \frac{ft^2}{2}$.

Suppose the time t to be divided into n equal parts; then since the force f is uniform it generates in each second a velocity f, and in the interval $\frac{t}{n}$ it generates a velocity $\frac{ft}{n}$. Consequently at the conclusions of the n portions of time into which we have supposed the whole time t divided, the velocity will be represented by the terms of the series,

$$\frac{ft}{n},\ \frac{2ft}{n},\ \frac{3ft}{n},\ldots\ldots\ldots\frac{(n-1)ft}{n},\ ft;$$

and at the commencement of those same portions of time the velocities will be

$$0, \frac{ft}{n}, \frac{2ft}{n} \ldots\ldots\ldots, \frac{(n-2)ft}{n}, \frac{(n-1)ft}{n}.$$

Now let us suppose that the body moves through each of the n intervals uniformly with the velocity which it has at the *conclusion* of the interval, and let s_1 be the number of feet described upon this hypothesis; then we have

$$s_1 = \frac{ft}{n} \times \frac{t}{n} + \frac{2ft}{n} \times \frac{t}{n} + \ldots\ldots\ldots + ft \times \frac{t}{n},$$

$$= \frac{ft^2}{n^2}(1 + 2 + \ldots\ldots\ldots + n),$$

$$= \frac{ft^2}{n^2} \times \frac{n(n+1)}{2} = \frac{ft^2}{2}\left(1 + \frac{1}{n}\right).$$

Again, let us suppose the body to move through each interval uniformly with the velocity which it has at the *commencement* of the interval, and let s_2 be the space described upon this hypothesis; then we have

$$s_2 = 0 \times \frac{t}{n} + \frac{ft}{n} \times \frac{t}{n} + \ldots\ldots\ldots + \frac{(n-1)ft}{n} \times \frac{t}{n},$$

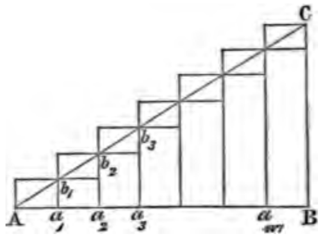
$$= \frac{ft^2}{n^2}\{0 + 1 + 2 + \ldots\ldots\ldots + (n-1)\},$$

$$= \frac{ft^2}{n^2} \times \frac{n(n-1)}{2} = \frac{ft^2}{2}\left(1 - \frac{1}{n}\right).$$

Now it is clear that the quantity s which we seek is less than s_1 and greater than s_2, because the body is in reality at any instant moving with less velocity than it has at the conclusion of one of the intervals and greater velocity than it has at the commencement; moreover the larger we take n, that is the smaller we make the intervals, the more nearly do s_1 and s_2 become equal to each other, and each of them to $\frac{ft^2}{2}$; consequently s which is always intermediate

to s_1 and s_2 must be equal to $\frac{ft^2}{2}$. $\therefore$ *If* s *be the number* &c. Q.E.D.

13. We will give another demonstration of this very important proposition. The demonstration which we now give is precisely the same in principle as the preceding, but is exhibited in a geometrical form.



Take a straight line AB, and divide it into n equal parts at the points $a_1\, a_2\, a_3$......And let the line AB represent a portion of *time*, and the lines $A\,a_1$, $a_1\,a_2$, $a_2\,a_3$, ...therefore represent each the nth part of the time represented by AB. At a_1 draw $a_1 b_1$ perpendicular to AB, and let $a_1\, b_1$ represent the *velocity* generated by the force f in the time $A a_1$; in like manner at $a_2\, a_3$,...draw $a_2\, b_2$, $a_3\, b_3$,...perpendicular to AB and representing the velocity generated in the times $A a_2$, $A a_3$,...; and let BC be the last of the lines so drawn, so that BC represents the velocity generated in the time AB. Now since the velocity generated by a uniform force is proportional to the time

$$a_1 b_1 : a_2 b_2 : a_3 b_3 : \ldots BC :: A a_1 : A a_2 : A a_3 : \ldots\ldots AB;$$

and consequently, if we join the points A and C, AC will pass through all the points $b_1\, b_2\, b_3$......Complete the rectangles, $A\, b_1$, $a_1\, b_2$, $a_2\, b_3$......, also $a_2\, b_1$, $a_3\, b_2$,......, as in the figure.

Then since the space described by a body moving uniformly is jointly proportional to the velocity and the time, if we suppose the body to move during each of the n intervals $A\, a_1$, $a_1\, a_2$, $a_2\, a_3$,......uniformly with the velocity which it has at the end of the intervals, the spaces described in those intervals will be represented by the rectangles $A\, b_1$, $a_1\, b_2$, $a_2\, b_3$,......; and the whole space described *in the time* AB upon this hypothesis will be represented by *the triangle* ABC + the n exterior triangles. In like manner,

if we suppose the body to move uniformly through each of the intervals with the velocity which it has at the commencement of the interval, the whole space described will be represented by the triangle ABC – the n interior triangles. It is evident that the sum of the areas of the triangles spoken of in these two cases is equal to half the rectangle $a_{n-1}\,C$; hence if s be the space actually described by the body, s will be intermediate in value to

$$ABC + \tfrac{1}{2}\,a_{n-1}\,C, \text{and } ABC - \tfrac{1}{2}\,a_{n-1}\,C:$$

and the greater number of intervals we take the more nearly does the motion upon either of the above hypotheses approach to the actual motion of the body, and the smaller does $a_{n-1}\,C$ become; hence it appears that s being always intermediate to the above two values, however many intervals there may be, must be equal to ABC.

Hence then the triangle ABC represents the space described. And if we denote the whole time by t as before, we have $AB = t$, $BC = ft$, (Art. 12. p. 28), and therefore

$$s = \frac{1}{2}t \times ft = \frac{ft^2}{2}.$$

We have thus arrived by a geometrical method at the same result as before; the student who examines the two processes carefully will perceive that the principle of them is precisely the same, it will not however on that account be less useful to him to study both.

14. It will now be seen, as was before stated, that there is a necessary connexion between the space through which a body falls in the first second, and the accelerating force acting upon it. For in the formula above proved make $t = 1$, then we have $f = 2s$; in other words, *the accelerating force is measured by twice the space described from rest in a unit of time,* or in one second. Thus, assuming gravity to be a uniform force, if we observe that a body falls through 16.1 feet in the first second of its fall, we shall know that the measure of the accelerating force of gravity is 32.2, or that $g = 32.2$.

15. Let us now take a few examples of the application of the formula $s = \frac{ft^2}{2}$.

Ex. 1. A body moves from rest with a uniformly accelerated motion, and it is found that at the end of 2 minutes 4 seconds it has passed over 63.4 feet; find the accelerating force.

In this case, $t = 124$ when $s = 63.4$;

$$\therefore f = \frac{2 \times 63.4}{(124)^2} = \frac{317}{38440} = .82 \text{ nearly};$$

in other words, the force is such as to add in each second a velocity of .82 feet per second.

Ex. 2. Two heavy bodies are let fall at an interval of one second; how far apart will they be at the end of three seconds from the fall of the first?

The space described by the first $= \frac{g}{2} \times 3^2 = \frac{9}{2} g$,

.... second $= \frac{g}{2} \times 2^2 = \frac{4}{2} g$;

$\therefore$ the interval between them $= \frac{5}{2} g = 5 \times 16.1 = 80.5$ feet.

Ex. 3. A stone requires two seconds to fall to the bottom of a well; find the depth.

$$\text{The depth} = \frac{g}{2} \times 2^2 = 2 \times g = 64.4 \text{ feet}.$$

Ex. 4. Generally, if n seconds be required for a stone to fall to the bottom of a well, the depth will be $n^2 \times 16.1$.

Ex. 5. Two bodies fall under the action of gravity from the same point but not at the same time; the distance between them is noted at two instants a second apart from each other; find the time which elapsed between the starting of the two bodies.

Let x be the number of seconds required, and let a be the distance between the bodies after n seconds from the fall of the last, b after $n + 1$ seconds; then a and b are the observed distances; and we have

$$a = \frac{g}{2}(n+x)^2 - \frac{g}{2} n^2 = \frac{g}{2}(2nx + x^2),$$

$$b = \frac{g}{2}(n+1+x)^2 - \frac{g}{2}(n+1)^2 = \frac{g}{2}\{2(n+1)x + x^2\};$$

$\therefore\ b - a = gx,$

and $x = \dfrac{b-a}{g}$, the number of seconds required.

Ex. 6. The distances through which a body is observed to move in successive seconds are in the proportion of the numbers 1, 3, 5, 7...; prove that the body is under the action of a uniform accelerating force.

Let s_n be the distance of the body from the starting-point after n seconds; then

$$\begin{aligned} s_2 - s_1 &= 3s_1, \\ s_3 - s_2 &= 5s_1, \\ s_4 - s_3 &= 7s_1, \\ \&c. &= \&c. \\ s_n - s_{n-1} &= (2n-1)s_1, \end{aligned}$$

$\therefore$ by addition

$$\begin{aligned} s_n &= s_1\{1 + 3 + 5 + \ldots + (2n-1)\} \\ &= s_1 n^2. \end{aligned}$$

Comparing this with the formula $s = \dfrac{ft^2}{2}$, we see that the body is under the action of an accelerating force measured by $2s_1$.

16. When a body moves from rest under the action of a uniform force f, we know that if v be the velocity at the time t, and s the space described, then $v = ft$, and $s = \dfrac{ft^2}{2}$; from these two relations we can determine what will be the velocity of the body when it has moved through a given space, or conversely, through what space it must move in order to acquire a certain velocity.

$$\text{For since } v = ft,\ \therefore\ v^2 = f^2t^2 = f \times ft^2 = 2fs.$$

Hence if s be given, v is known; and conversely, from the equation $s = \dfrac{v^2}{2f}$, we know the value of s corresponding to a given value of v. The student will do well to notice

that in the case of a uniform force the velocity $\propto$ the time, and $\propto$ the square root of the space.

In the case of a body falling under the action of gravity, we have $v^2 = 2gh$, and it is usual to speak of v as the velocity *due to the height h.*

17. The application of the formulæ of the preceding article will be seen from some Examples.

Ex. 1. A ball is thrown into the air with a velocity which would carry it through 10 feet in one second; find how high it will rise.

It will rise through such a space as shall enable gravity to destroy a velocity of 10 feet per second; and the space required for this is the same as would be required to generate the same velocity in a body falling from rest; hence

$$\text{height required} = \frac{10^2}{2g} = \frac{100}{2 \times 32.2} = \frac{25}{16.1} = 1.55 \text{ feet.}$$

Ex. 2. A well is 96 feet deep; find the velocity with which a falling body will strike the bottom of the well.

Let v be the velocity;

$$\text{then } v^2 = 2g \times 96 = 6182.4;$$
$$\therefore\ v = 78.6 \text{ feet per second.}$$

Ex. 3. A body falls through a distance of 100 feet; compare the velocities at the middle and end of the fall.

Let v, v' be the two velocities.

$$\text{Then } \frac{v^2}{v'^2} = \frac{50}{100} = \frac{1}{2};$$
$$\therefore\ \frac{v}{v'} = \frac{1}{\sqrt{2}} = \frac{1}{1.414} = \frac{1000}{1414} \text{ nearly.}$$

Ex. 4. Through what height must a body fall to acquire the velocity ng?

Let x be the height; then

$$(ng)^2 = 2gx;$$
$$\therefore\ x = \frac{n^2}{2}g.$$

18. Hitherto we have considered only the case of a *body falling* from rest, or moving from rest under the

action of a uniform accelerating force. We must now extend our conclusions to the case of a body which starts with a certain given velocity, and which is then acted upon by a force; for instance, suppose that instead of letting a ball fall towards the earth, we throw it vertically downward, so that when it leaves the hand, which is the instant from which we measure our time, it has a given velocity as V. It is clear that in this case there is no new principle involved, for the motion will be precisely the same as that of a body which has been previously falling during a sufficient time to enable gravity to generate a velocity V, that is, during the time $\frac{V}{g}$. We may, in fact, solve the problem upon this principle; we have in general for the space described in time t,

$$s = \frac{gt^2}{2};$$

in the present case the space described will be the same as if the body had been falling through the time $t + \frac{V}{g}$, omitting the distance through which it has fallen during the said interval $\frac{V}{g}$; now in the time $\frac{V}{g}$ the body falls through $\frac{g}{2} \times \frac{V^2}{g^2} = \frac{V^2}{2g}$, hence we shall have

$$s = \frac{g}{2}\left(t + \frac{V}{g}\right)^2 - \frac{V^2}{2g},$$

$$= Vt + \frac{gt^2}{2}.$$

In this formula we observe that Vt is the space through which the body would have fallen had there been no force, and $\frac{gt^2}{2}$ is the space through which it would have fallen had there been no initial velocity; the actual space

described is the sum of these two, as indeed might have been anticipated; and this consideration does, in fact, lead to the simplest mode of treating the problem, which is as follows.

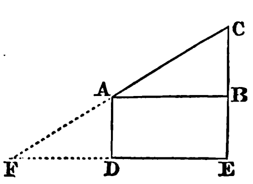
19. PROP. *Let a body start with a velocity* V, *and let it be acted upon by a uniform accelerating force* f, *and let* v *be the velocity, and* s *the space passed through, at the time* t; *then will* $v = V + ft$, *and* $s = Vt + \frac{ft^2}{2}$.

Uniform accelerating force adds equal amounts of velocity in equal times; hence if when the motion commences the velocity be V, and in each second a velocity f be added, it follows that after t seconds the velocity is $V + ft$; in other words, $v = V + ft$.

Again, if no force acted, the body would in t seconds pass over a space Vt. Also the force is such as to be capable of carrying the body during the same time over the space $\frac{ft^2}{2}$; but the space through which the force will carry it cannot be affected by the velocity which it had at the commencement of the action of the force, since the effect of force is always to add equal velocities in equal times, whatever be the previous velocity; hence the space $\frac{ft^2}{2}$ must be added to the space which would be described without the action of f, in other words

$$s = Vt + \frac{ft^2}{2}.$$

20. It may be remarked that the figure which will represent s in this case, according to the method of Art. 13, will be a figure composed of a rectangle and triangle.



Let, as before, AB represent the *time, and* BC the velocity generated

in that time; join AC, and on the opposite side of AB construct the rectangle $ADEB$, AD or BE representing the velocity V; then the figure $ADEC$ represents the space described.

And the method by which we first arrived at the formula $s = Vt + \frac{gt^2}{2}$, (Art. 18), may be illustrated by the figure, if we produce CA, ED to meet in F. For then FD represents the time during which the body must fall in order to acquire the velocity V or AD; and the whole space described in the time AB or DE will be represented by the triangle FCE − triangle FAD.

21. We will now subjoin some examples of the preceding formulæ.

Ex. 1. A bullet is fired upwards with a velocity of 1500 feet per second; find how high it will rise, and the time which elapses before it again reaches the ground.

If $2t$ be the time required, and x the height, we have

$$0 = 1500 - gt$$

since the body stops when $v = 0$;

$$\therefore\ t = \frac{1500}{g} = 46.5 \text{ seconds,}$$

$$\text{and } 2t = 93,$$

$$x = 1500 \times \frac{1500}{g} - \frac{g}{2}\left(\frac{1500}{g}\right)^2$$

$$= \frac{1500^2}{2g} = 34875 \text{ feet.}$$

Ex. 2. A body is projected downwards, and at the end of n seconds it is observed to be moving with a given velocity V; find the velocity of projection.

Let v be the velocity of projection;

$$\text{then}\quad V = v + ng;$$

$$\therefore\ v = V - ng.$$

Ex. 3. With what velocity must a stone be projected, in order that it may descend to the bottom of a well 100 feet deep in one second?

Let V be the velocity; then we must have

$$100 = V + \frac{g}{2},$$

$$\text{or } V = 100 - 16.1 = 83.9.$$

Ex. 4. A body is projected upwards with a velocity $3g$, what will be its velocity and what its position at the end of 10 seconds?

It will require 3 seconds to destroy the velocity $3g$; the body will then descend, and at the end of 7 seconds more its velocity will be $7g$; and it will have fallen through $\frac{g}{2} 7^2$ feet, hence the body will be found $\frac{g}{2} 7^2 - \frac{g}{2} 3^2$, or $20g$ feet below the point from which it started.

Ex. 5. A body is let fall from the height of 100 feet, and at the same moment a body is projected vertically upwards from the ground; they meet halfway; what was the velocity of projection of the second body?

Let t be the time which elapses before they meet; and V the velocity required; then

$$50 = \frac{gt^2}{2},$$

$$\text{and } 50 = Vt - \frac{gt^2}{2};$$

$$\therefore\ 100 = Vt,$$

$$\text{and } 50 = \frac{g}{2}\left(\frac{100}{V}\right)^2;$$

$$\therefore\ V^2 = 100\,g = 3220,$$

$$V = 57 \text{ nearly.}$$

22. As when a body starts from rest we have the formula $v^2 = 2fs$ (Art. 16), so when a body starts with a given velocity V, if v be the velocity when it has passed over a space s, we shall have $v^2 = V^2 + 2fs$.

For we have the two formulæ

$$v = V + ft,$$

$$\text{and } s = Vt + \frac{ft^2}{2}.$$

$$\text{Hence}\quad v^2 = (V + ft)^2$$
$$= V^2 + 2Vft + f^2t^2$$
$$= V^2 + 2f\left\{Vt + \frac{ft^2}{2}\right\}$$
$$= V^2 + 2fs.$$

23. This formula we might also have deduced in a manner analogous to that adopted in Art. 18. For let h be the space through which the body must move in order to acquire the velocity V; then when it has moved through s starting with the velocity V, the circumstances will be the same as if it had moved through $h + s$ starting from rest:

$$\therefore\ v^2 = 2f(h + s) = 2fh + 2fs$$
$$= V^2 + 2fs, \text{ since } h \text{ is such that } V^2 = 2fh.$$

24. Ex. 1. A body is projected upwards with a velocity of 100 feet per second; with what velocity will it pass a point 2 feet from the ground?

Let v be the velocity required; then

$$v^2 = 100^2 - 2g \times 2$$
$$= 10000 - 128.8 = 9871.2;$$
$$\therefore\ v = 99.3.$$

Ex. 2. A body is projected upwards with a velocity ng, find its velocity at a height gx.

We have
$$v^2 = n^2g^2 - 2g \times gx,$$
$$= (n^2 - 2x)g^2.$$

Hence the body will stop when

$$x = \frac{n^2}{2}.$$

Ex. 3. A body is projected upwards with a velocity V, find the position of the body when it is descending with a velocity v.

The height to which the body will rise is $\frac{V^2}{2g}$. Let x be the distance through which it must fall to acquire the velocity v; then $x = \frac{v^2}{2g}$; therefore the distance required measured from the starting-point is $\frac{v^2 - V^2}{2g}$.

25. The preceding articles contain all the formulæ

necessary for solving problems of falling bodies, or more generally of the rectilinear motion of a particle under the action of a uniform accelerating force. For convenience we will here collect them under one view;

If the body start from rest,

$$\left.\begin{aligned} v &= ft, \\ v &= \frac{ft^2}{2}, \\ v &= 2fs. \end{aligned}\right\}$$

If the body start with a velocity V,

$$\left.\begin{aligned} v &= V + ft, \\ s &= Vt + \frac{ft^2}{2}, \\ v^2 &= V^2 + 2fs. \end{aligned}\right\}$$

26. There is a slight modification of the problem of falling bodies which may be treated of in this place; and that is the case of a body falling under the action of gravity upon an inclined plane. If we denote the angle of inclination of the plane by a, and resolve the accelerating force of gravity at the time t into two portions, one in the direction of the plane, the other in the direction perpendicular to the plane, the former alone will accelerate the body's motion, the latter will cause a pressure upon the plane. The resolved part in the direction of the plane will be $g \sin a$, and this being a uniform force all our formulæ will apply by taking $g \sin a$ in the place of that which we have called in general f. We shall say nothing at present concerning the pressure upon the plane caused by the resolved part $g \cos a$, as this would lead us into considerations for which the student is not prepared; it will be sufficient for him to perceive that the motion upon the inclined plane will be uniformly accelerated, but in a less degree than if the body fell freely in the ratio of $\sin a : 1$. For instance, if $a = 30^\circ$, the velocity *generated* upon such a plane in a given time will be only *half what* would be generated if the body fell freely.

One consequence follows from what we have just said,

which is worthy of notice, and is contained in the following proposition.

27. Prop. *The velocity acquired by a body in falling down a given inclined plane is the same as that of a body which has fallen freely through the same vertical distance.*

Let l be the length of the inclined plane, α the angle of inclination, v the velocity acquired, x the distance through which the body must fall vertically in order to acquire the same velocity. Then by what precedes,

$$v^2 = 2g \sin \alpha \times l;$$

also, by hypothesis,

$$v^2 = 2gx;$$

$$\therefore \; x = l \sin \alpha;$$

and $l \sin \alpha$ is the vertical distance through which the body has fallen. Hence *the velocity* &c. Q.E.D.

28. It is hardly necessary to state that the preceding method of treating the problem of a body falling on an inclined plane applies to all cases of bodies moving upon planes inclined at an angle to the direction of an uniform force. There is, however, hardly any case with which we are likely to be concerned besides that of a heavy body upon an inclined plane, and we have therefore spoken more particularly concerning that case.

We have supposed the inclined plane to be perfectly smooth, so that no velocity can be destroyed by friction. Practically, of course, this is a condition which cannot be realized; and experiment teaches us that friction produces a retarding force proportional to the pressure upon the plane; the pressure upon the plane being constant, there will be an uniform retardation due to friction. As, however, this subject would lead us to the consideration of the measure of the pressure upon the plane, we shall not pursue it further at present.

29. A few examples of motion upon an inclined plane shall conclude this chapter.

Ex. 1. *A body is observed to fall down an inclined plane 3 feet long in 9 seconds; find the inclination of the plane.*

Let a be the inclination; then

$$3 = g \sin a \times \frac{81}{2};$$

$$\therefore \sin a = \frac{1}{27 \times 16.1} = \frac{1}{434.7}.$$

Ex. 2. A body falls down an inclined plane 6 feet long, inclination 30°; find the velocity at the bottom of the plane.

We have $\qquad v^2 = 2g \times 6 \sin 30^\circ = 6g = 193.2,$

$$v = 13.9 \text{ nearly.}$$

Ex. 3. Through what length of a plane inclined at an angle of 45° must a body fall to acquire a velocity $6g$?

Let x be the length required;

We have $(6g)^2 = 2gx \sin 45^\circ$,

$$x = \frac{18g}{\sin 45^\circ} = 18\sqrt{2}g$$

$$= 497.55 \text{ feet.}$$

Ex. 4. A body is projected upwards, along an inclined plane, with a given velocity, find how high it will ascend, and the time of ascent.

If v be the velocity at the time t, when it has ascended through a space s, a the angle of the plane, and V the given velocity of projection, we have

$$v^2 = V^2 - 2gs \sin a;$$

when $v = 0$ the body will stop, and the distance required will be given by the equation,

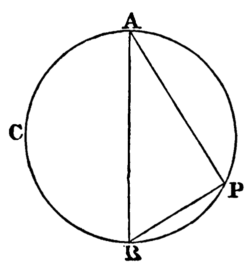
$$s = \frac{V^2}{2g \sin a}.$$

For the time, we have,

$$v = V - gt \sin a,$$

and the body stops when

$$t = \frac{V}{g \sin a}.$$



Ex. 5. Let ACB be a circle in a vertical plane, A its highest point; the time of descent down all chords drawn through A, considered as inclined planes, will be the same.

Let AP be any chord, a its inclination to the horizon; draw the vertical diameter AB, and join BP, then $ABP = 90^\circ - BAP = a$: now if t be the time of descent down AP, we shall have

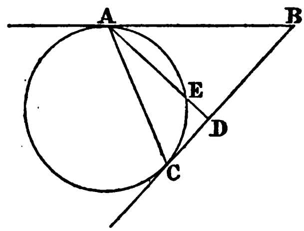
$$AP = g \sin \alpha \frac{t^2}{2};$$

$$\text{but, } AP = AB \sin \alpha,$$

$$\therefore AB = \frac{gt^2}{2},$$

and $t = \sqrt{\dfrac{2\,AB}{g}}$, which is independent of α, and is therefore the same for all chords.

Ex. 6. From a given point to draw the line, down which as an inclined plane a particle will descend to a fixed line in the shortest time possible.



Let A be the given point, and through it draw a horizontal line to meet the given line in B; in the given line take BC equal to BA; join AC, which shall be the line required.

This will easily appear from the fact that a vertical circle having A for its highest point will *touch* the given straight line in C; for if AC be not the line of quickest descent, let AED cutting the circle just mentioned in D be the required line; then the time of descent down AC is equal to the time down AE, and therefore is less than the time down AD. Therefore AC is the line of quickest descent.

And it may be shewn, that in general the line of quickest descent from a point to any given curve will be found, by describing a circle, to touch the horizontal line passing through the given point at that point, and also to touch the given curve. The chord joining the points of contact will be the line of quickest descent.

CONVERSATION UPON THE PRECEDING CHAPTER.

P. It is stated that the First Law of Motion is the result of the labour of centuries; when was the subject of Dynamics first discussed?

T. Aristotle made attempts to solve problems of motion, but met with no success; or perhaps we may say that he met with worse than none, for the influence which

he enjoyed during the middle ages served to stereotype his errors, and thus to retard the progress of science in Europe. How it happened that Archimedes, who exhibited so great a genius for mechanics, entirely failed in overcoming the difficulties which beset the science of Dynamics it may be hard to say; but this is certain, that he would obtain no guidance from the study of Aristotle, and would perhaps find his difficulties considerably increased by what Aristotle had done. And it must never be forgotten that great difficulties did lie in the way of forming a science of Dynamics, on account of the manner in which every case of motion upon the earth is complicated by the introduction of disturbing forces, such as friction, the resistance of the air, and the like.

The fact that all bodies come to rest unless some force acts upon them, and that heavy bodies fall more rapidly than lighter ones, would almost infallibly suggest wrong conceptions of force to those who first considered such problems. As for the views of Aristotle, it is quite clear that they never could lead to any real progress in science; it was not that he did not observe, but that he accounted for the results of his observation in an erroneous manner; instead of considering motion as a property of bodies produced by external agents, he considered it as something belonging to the bodies themselves, and instead of referring to *force* as that which changes the state of bodies, he referred to certain general dogmas of his own, deduced from principles totally foreign to Dynamics. Thus motion was arbitrarily divided into two classes, *natural* and *violent;* the natural gradually increased, becoming quicker and quicker, while the violent diminished and ultimately vanished entirely; for instance, a body thrown along the earth's surface moves with a continually diminishing velocity until it come to rest, and this was accounted for by saying that the motion was violent; whereas a heavy body let fall moves with a continually accelerated velocity, and this was accounted for by saying that the motion was natural. Besides this there was the fundamental error, which I

alluded to before, of bodies falling more quickly in proportion as they are heavier.

P. I am not surprised at that error obtaining, for I feel great difficulty in divesting my own mind of the notion; yet the experiment upon the guinea and feather under the exhausted receiver of an air-pump makes it quite clear that the notion is erroneous.

T. Not only so; you may convince yourself as to the side upon which the truth lies, without any air-pump. Suppose we let fall a body of certain magnitude, a brick for instance; it falls to the earth in a certain manner; let it again fall, and let another brick, exactly resembling it, be let fall by the side of the first, it is clear that the two will fall exactly in the same manner, and therefore will continue side by side during their fall; the same will take place if we let fall a third and a fourth; that is, however many bricks we cause to fall in contact with each other, they will all remain in contact during the fall; this being the case, we may conceive them to be all fastened together, for since they do not separate when free, the problem will not be altered by supposing them fastened. This proves then, that one brick and a mass consisting of a thousand will fall to the ground precisely in the same manner, and in the same time from a given height.

P. This view seems to shew that it ought be be so, but still I feel a difficulty in realising the truth.

T. I have no doubt but that you will understand the matter better when you have read Chapter IV; but even now you must remember that the earth attracts all bodies, and that it is reasonable to suppose that it attracts two bricks with twice the force with which it attracts one, and that as there is twice the work to be done, namely, two bricks to be moved instead of one, it is not difficult to believe that the time of doing the work will be the same.

P. That remark seems to throw some light upon the difficulty; and I must now apologize for interrupting you in speaking of Aristotle.

T. I have little more to say upon his dynamical attempts; all that I wish you to bear in mind is, that he and those who followed him failed in making any advance, chiefly because they had not a clear idea of matter as inert and changing its motion only in consequence of an external cause which we call *force;* they had not the first law of motion, in other words they were ignorant of the *inertia* of matter, and this was a bar to all progress.

P. Do you speak of the *inertia* of matter and the first law of motion as the same thing?

T. Yes; the property of matter expressed by the first law of motion is frequently called its inertia, and the name is sufficiently significant; it may however perhaps give rise to a wrong conception which I will anticipate. When a body is at rest it does not offer any real resistance to motion in consequence of its inertia, that is, there is no force so small as not to be able to put it in motion; of course the rapidity with which velocity will be generated depends upon the intensity of the force, and if the force be extremely small, velocity will be generated very gradually, but however small it may be the inertia of the body will not prevent it from moving. All that is meant by inertia is, that there is no internal cause of change.

P. You have stated that Aristotle and his followers failed to form a true system of Dynamics; to whom are we indebted for true dynamical views?

T. The glory of establishing dynamics upon a true basis belongs to Galileo, who was born in 1564, and died in 1642; some few previous writers had shewn more or less of knowledge of the true principles, but Galileo was the first who gave a complete and satisfactory exposition. This he did in his celebrated dialogues; three interlocutors, Salviati, Sagredo, and Simplicio, are supposed to discuss Galileo's Treatises on various mechanical subjects, and in the course of their conversation the various difficulties which can be raised are argued. The third Dialogue entitled,

De Motu Locali, is that in which the principles of uniform and uniformly accelerated motion are explained, and as it will be interesting to you to see how the subject presented itself to the mind of the first great discoverer, and it will also tend to impress the principles upon your mind to see them in a rather different form, I will give you a sketch of the dialogue.

The first portion of the dialogue is entitled "De Motu Æquabili." Galileo defines æquable motion thus. "Æquabilem, seu uniformem motum intelligo eum, cujus partes quibuscunque temporibus æqualibus peractæ, sunt inter se æquales."

To this definition he appends the following four axioms.

I. Spatium transactum tempore longiore in eodem motu æquabili majus est spatio transacto tempore breviori.

II. Tempus quo majus spatium conficitur, in eodem motu æquabili longius est tempore, quo conficitur spatium minus.

III. Spatium a majori velocitate confectum tempore eodem majus est spatio confecto a minori velocitate.

IV. Velocitas, qua tempore eodem conficitur majus spatium, major est velocitate, qua conficitur spatium minus.

And then follow these six Theorems.

I. Si mobile æquabiliter latum, eademque cum velocitate duo pertranseat spatia, tempora lationum erunt inter se ut spatia peracta.

That is to say, if the velocity be constant the space described will vary as the time.

II. Si mobile temporibus æqualibus duo pertranseat spatia, erunt ipsa spatia inter se ut velocitates. Et si spatia sint ut velocitates tempora erunt æqualia.

That is to say, if the time be constant the space will vary as the velocity, and if the space vary as the velocity, the time must be constant.

III. In æqualibus velocitatibus per idem spatium latorum tempora velocitatibus è contrario respondent.

That is to say, if the space be constant, the time will vary inversely as the velocity.

IV. Si duo mobilia ferantur motu æquabili, inæquali tamen velocitate; spatia, temporibus inequalibus ab ipsis peracta, habebunt rationem compositam ex ratione velocitatum et ex ratione temporum.

That is to say, if the velocity and time both vary, the space will vary as the two jointly; or according to our notation, $s \propto vt$.

V. Si duo mobilia æquabili motu ferantur, sint tamen velocitates inæquales et inæqualia spatia peracta, ratio temporum composita erit ex ratione spatiorum, et ex ratione velocitatum contrarie sumptarum.

That is to say, $t \propto \frac{s}{v}$.

VI. Si duo mobilia æquabili motu ferantur, ratio velocitatum ipsarum composita erit ex ratione spatiorum peractorum, et ratione temporum contrarie sumptorum.

That is to say, $v \propto \frac{s}{t}$.

Thus it will be seen that the whole first part of the Dialogue is equivalent to the establishment of the formula $s = vt$. The second, which contains what is called "Piu sottile e nuova contemplazione," is entitled "De Motu naturaliter accelerato."

In this section Galileo treats of falling bodies, the laws of which he deduces correctly from the supposition of their motion being *uniformly accelerated.* The manner in which he arrives at the conclusion that the acceleration must be uniform, and the manner in which he defines uniformity are deserving of attention.

Modo de motu accelerato pertractandum. Et primo definitionem ei, quo utitur natura, apprime congruentem *investigare* atque explicare convenit. Quamvis enim ali-

quam lationis speciem ex arbitrio confingere, et consequentes ejus passiones contemplari non sit inconveniens, (ita enim, qui Helicas, aut Conchoides lineas ex motibus quibusdam exortas, licet talibus non utatur natura, sibi finxerunt, earum symptomata demonstrarunt cum laude) tamen quandoquidem quadam accelerationis specie gravium descendentium utitur natura, eorundum speculari passiones decrevimus, si eam, quam allaturi fuimus de nostro motu accelerato definitionem, cum essentia motus naturaliter accelerati congruere contigerit. Quod tandem post diuturnas mentis agitationes reperisse confidimus, ea potissimum ducti ratione, quia symptomatis deinceps a nobis demonstratis apprime respondere, atque congruere videatur ea, quæ naturalia experimenta sensui representant. Postremo ad investigationem motus naturaliter accelerati nos quasi manu duxit animadversio consuetudinis atque instituti ipsiusmet naturæ in cæteris suis operibus omnibus, in quibus exercendis uti consuevit mediis primis, simplicissimis, facilimis: neminem enim esse arbitror, qui credat natatum aut volatum simpliciori, aut faciliori modo exerceri posse, quam eo ipso, quo pisces et aves instinctu naturali utuntur. Dum igitur lapidem ex sublimi a quiete descendentem nova deinceps velocitatis acquirere incrementa animadverto, cur talia additamenta simplicissima, atque omnibus magis obvia ratione fieri non credem? Quod si attente inspiciamus, nullum additamentum, nullum incrementum magis simplex inveniemus, quam illud, quod semper eodem modo superaddit. Quod facile intelligimus maximam temporis, atque motus affinitatem inspicientes: sicut enim motus æquabilitas, et uniformitas per temporum, spatiorumque æquabilitates definitur atque concipitur, (lationem enim tunc æquabilem appellamus cum temporibus æqualibus æqualia conficiuntur spatia) ita per easdem æqualitates partium temporis, incrementa celeritatis simpliciter facta percipere possumus; mente concipientes motum illum uniformiter, eodemque modo continue acceleratum esse, dum temporibus quibuscunque æqualibus æqualia ei superadduntur celeritatis additamenta. Adeo ut sumptis quotcunque temporis particulis æqualibus a primo instanti

C

in quo mobile recedit a quiete, et descensum aggreditur, celeritatis gradus in prima cum secunda temporis particula acquisitus duplus sit gradus, quem acquisivit mobile in prima particula: gradus vero quem obtinet in tribus particulis, triplus, quem in quatuor, quadruplus ejusdem gradus primi temporis. Ita ut (clarioris intelligentiæ causa) si mobile lationem suam continuaret juxta gradum, seu momentum velocitatis in prima temporis particula acquisitæ, motumque suum deinceps æquabiliter cum tali gradu extenderet, latio hæc duplo esset tardior ea, quam juxta gradum velocitatis in duabus temporis particulis acquisitæ obtineret; et sic a recta ratione absonum nequaquam esse videtur, si accipiamus intentionem velocitatis fieri juxta temporis extensionem: ex quo definitio motus, de quo acturi sumus, talis accipi potest: Motum æquabiliter, seu uniformiter acceleratum dico illum, qui a quiete recedens, temporibus æqualibus æqualia celeritatis momenta sibi superaddit.

The difficulties which are suggested by this view of the motion of falling bodies are discussed by the interlocutors; the reasoning by which Galileo arrives at his conclusion is manifestly faulty; nothing can be more loose and unsatisfactory than to base a principle upon its supposed simplicity and the axiom that nature will choose the simplest course, although this is not the only instance in which such reasoning has been rewarded by the discovery of truth. Moreover, in the case of the uniform acceleration of falling bodies, the method is particularly unsatisfactory, inasmuch as the result is only approximately true, the actual law of nature being in fact according to Galileo's estimate anything but the simplest. I think, however, it may fairly be doubted whether Galileo himself placed much stress upon his own reasoning, seeing that he is very careful to bring his views to the test of experiment: it seems to me very possible, that he may have been led by his experiments to the apprehension of the law, and may then have given *à priori* reasons for it in accordance with the spirit of the ancient philosophy.

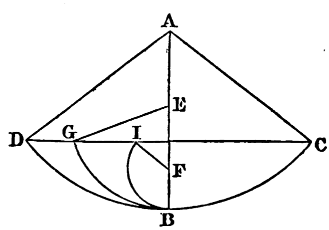
We now come to the grounds of the definition which was given of uniformly accelerated motion, and it is stated, that in order to make good this definition, the following principle must be conceded.

Accipio, says the author, gradus velocitatis ejusdem mobilis super diversas planorum inclinationes acquisitos tunc esse æquales, cum eorundem planorum elevationes æquales sunt.

This is equivalent to the proposition which has been proved in Art. 27, p. 41.

On this principle being propounded, one of the interlocutors, Sagredo, remarks, "It seems to me that such a supposition carries with it so much probability that it deserves to be at once admitted as true without controversy." Galileo, however, is not satisfied with this mode of establishing his very important theorem, but in the person of Salviati, proposes this simple and ingenious experiment.

Upon a wall trace a vertical line *AB* and a horizontal line *CD*: insert a nail at *A*, and from it suspend a heavy leaden ball by a fine string; it will hang in the line *AB*; raise the ball to *C*, the string being kept stretched; let go the ball and it will be found to describe a circular arc, and rise as nearly as



may be to the point *D* in the horizontal line *CD*, so nearly that we may fairly attribute any defect of such ascent to be due to the resistance of the air or of the string. Now insert a nail at *E* or at *F*, some point in the vertical *AB*; raise the ball as before and let it go; it will be found to rise to *G* or *I*, that is, still to a point in the horizontal line *CD*. From these experiments it is easy to conclude that the velocity acquired in falling through a circular arc is quite independent of the length of the arc, and dependent only upon the vertical height through which the body falls.

This reasoning appears so conclusive, that Sagredo thinks the postulate ought to be conceded as if it were

strictly demonstrated. Salviati, however, not wishing to make the experiment go for more than it is worth, points out the difference between the two cases, of motion on a plane and motion in a circular arc, and concludes by requesting that the postulate may be granted conditionally, in order that, having deduced other conclusions by means of it, they may be compared with experiment, and its truth be thus put to a more severe test.

Then follow thirty-eight propositions, partly problems and partly theorems, being principally such as would find a place in an elementary treatise on falling bodies in the present day, but of course demonstrated geometrically. I give you a few specimens.

THEOR. II. PROP. II. Si aliquod mobile motu uniformiter accelerato descendat ex quiete; spatia quibuscunque temporibus ab ipso peracta, sunt inter se in duplicata ratione eorundem temporum: nempe ut eorundem tempora quadrata.

From this he draws the corollary, that the spaces described by a body falling from rest in the first, second, third, ... seconds of its fall are as the odd numbers 1, 3, 5, ...

THEOR. VI. PROP. VI. Si a punctu sublimi, vel imo circuli ad horizontem erecti ducantur quælibet plana usque ad circumferentiam inclinata, tempora descensuum per ipsa erunt æqualia.

THEOR. XXI. PROP. XXXII. Si in horizonte sumantur duo puncta, et ab altero ipsorum quælibet linea versus alterum inclinetur, ex quo ad inclinatam recta linea ducatur, ex ea partem abscindens æqualem ei, quæ inter puncta horizontis intercipitur, casus per hanc ductam citius absolvetur, quam per quascunque alias rectas ex eodem puncto ad eandem inclinatam protractas. In aliis autem, quæ per angulos æquales hinc inde ab hac distiterint, casus fiunt temporibus inter se æqualibus.

PROB. XV. PROP. XXXVII. Dato perpendiculo, et plano inclinato, quorum eadem sit elevatio: partem in inclinato *reperire,* quæ sit æqualis perpendiculo, et conficiatur eodem *tempore* ac ipsum perpendiculum.

These will serve as a sample of the propositions enunciated and proved in this dialogue; at the close of them Sagredo expresses his admiration of the manner in which so many beautiful results flow from one simple principle, and his surprise that such results had escaped such men as Archimedes, Apollonius, and Euclid.

We will now, if you please, dismiss this dialogue of Galileo. Have you any further questions to ask?

P. Only one. I am somewhat puzzled by the geometrical proof of the formula $s = \frac{ft^2}{2}$ in Art. 13, in which a straight line represents time, another straight line velocity, and a parallelogram represents distance.

T. There is no real difficulty if you consider the principle upon which we represent quantities geometrically; all that is meant by taking a straight line to represent a certain period of time is, that the straight line contains as many units of length as the period of time contains units of time, an inch to a second, for example; and in like manner, if another straight line be taken to represent a velocity, it is intended that the straight line bears the same proportion to a standard line that the velocity in question does to a standard velocity; and a parallelogram may represent linear space in the same conventional manner, it being understood that the parallelogram contains as many units of square measure as the distance which it represents contains of linear measure. Sometimes the direction of straight lines is made use of as well as their magnitude to indicate the direction as well as the magnitude of a velocity or a force, but in the present instance it is the magnitude alone with which we are concerned, and this being so, parallelograms or cubes may be taken to represent distances, or, if we wished it, velocities or forces, just as well as straight lines.

EXAMINATION ON CHAPTER II.

1. Enunciate the First Law of Motion.

2. Distinguish between accelerating and moving force.

3. How is accelerating force measured?

4. What is meant by saying that 32.2 feet measures the force of gravity?

5. If 1 mile were taken as the unit of length, and 1 hour as the unit of time, what would be the measure of the earth's accelerating force?

6. Supposing the earth's accelerating force to be measured by 100, and 1 foot to be the unit of space, what would be the implied unit of time?

7. Investigate the formula $s = \frac{ft^2}{2}$, where f is a uniform accelerating force.

8. How may the accelerating force of gravity be determined, by letting fall a stone from the top of a tower? What objections would there be to this method?

9. If v be the velocity generated in the time t by a uniform accelerating force f, then $v = ft$.

10. The space described by a body under the action of an uniform accelerating force in a given time is half of that which would have been described if the body had moved uniformly during the time with the velocity which it has at the end of the time.

11. If a body acquire the velocity v in falling through a space s, then $v^2 = 2gs$.

12. If a body be projected with the velocity V and be acted upon by a uniform accelerating force f, find the velocity at the time t.

13. A body is projected upwards with a velocity V, find how high it will ascend.

14. Find also how long a time will elapse before the body again strikes the ground.

15. Prove and explain the formula

$$v^2 = V^2 + 2fs.$$

16. Explain the method of determining the motion of a heavy body upon an inclined plane.

17. The velocity acquired in falling down an inclined plane is that due to the vertical height.

18. A body is projected upwards with a velocity of 10 feet per second upon a plane inclined at 30^0 to the horizon; find how high it will ascend, and the time which elapses before it returns to the starting point.

19. The time of descent down all chords of a vertical circle from the highest point is the same.

20. From a given point find the plane of quickest descent to a vertical plane at a given distance from the point.

21. Compare the time of descent in the preceding problem with the time down a plane of twice the length.

22. Find the velocity with which a body must be projected upwards from the foot of a tower 50 feet high, so as to meet another body, let fall at the same time from the top of the tower, at a distance of 30 feet from the ground.

23. Through what height must a body fall to acquire a velocity of 1000 feet per minute?

24. A balloon is ascending vertically with a velocity V, and a stone let fall from it reaches the ground in n seconds: find the height of the balloon when the stone strikes the ground.

25. A body is observed to fall the last a feet of its descent in t seconds: find the height from which it fell.

26. A body has fallen through a distance of two miles, find the distance through which it fell in the last second.

27. The space described by a body in the fifth second of its fall is observed to be to the space described in the last second but four as $1:6$; what is the whole space described by the body?

28. A body falls through a feet at two places on the earth's surface; and it is observed that the time of falling is n seconds less, and the velocity acquired m feet greater at one place than at the other: compare the force of gravity at the two places.

29. Two bodies are projected at the same instant towards each other, from the extremities of a vertical line, each with the velocity which would be acquired in falling down it. Where will they meet?

30. A body being projected down an inclined plane with the velocity which would be acquired in falling down its perpendicular height,

the time of descent is found to be that of falling down the height. Required the plane's inclination.

31. Determine that diameter of a vertical circle, down the latter half of which a body falls in the same time as down the whole vertical diameter.

32. Determine that point in the hypothenuse of a right-angled triangle, having its base parallel to the horizon, from which the time of a body's descent down an inclined plane to the right angle is least.

CHAPTER III.

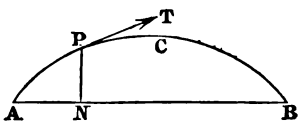
ON THE MOTION OF A PARTICLE UNDER THE ACTION OF AN ACCELERATING FORCE, WHEN THE MOTION IS NOT RECTILINEAR. SECOND LAW OF MOTION. PROJECTILES.

1. WE have seen that when a body is acted upon by no external force it will either be at rest or will move uniformly in a straight line; and if a body be moving in a straight line and a force act upon it in the direction of that straight line, the body's motion will be accelerated or retarded as the case may be, but the rectilinear character of the motion will not be destroyed. The most obvious example of this kind of motion is that of falling bodies, and this is the problem with which we have been chiefly concerned in the preceding chapter: we have now to enter upon a more general problem, namely, the motion of a body when influenced by a force which does not act in the direction of the body's motion, and under the action of which therefore the motion cannot be rectilinear.

The kind of motion referred to will be understood best by reference to an example. As in the former chapter our chief problem was that of falling bodies, so in this our chief problem will be that of bodies which fall under the action of gravity but not in a straight line, the motion for instance of a ball thrown in any direction. A body so thrown is called a *projectile*, and it is a definite dynamical problem to determine all the circumstances of the motion. We must observe however that any theoretical conclusions to which we may come concerning the motion of projectiles will not be experimentally correct, because in practice bodies move through the air which resists and modifies their motion, whereas we shall suppose bodies to move as though there were no air, that is, in a vacuum: the actual

determination of the motion of a projectile, taking account of the resistance of the air, is a very important problem in consequence of its application to gunnery, and at the same time a very difficult one; the theoretical case of a body moving in a vacuum is very much more simple, and will serve our purpose equally well as an illustration of dynamical principles.

2. Let us then consider what takes place when a body is thrown in any direction, not coinciding with the vertical direction; the body will describe a curve, and having risen to a certain height will then again descend and strike the ground. This curve will evidently lie all in one plane, since there will be nothing to draw it out of the vertical plane in which it begins to move. Let then ACB be the path of the body; A being the point from which it starts, B the point at which it strikes the ground, C the highest point of the path.



Now at any instant the direction of the body's motion is the same as that of the tangent of the curve in which it is moving: thus if P be any position of the body, PT the straight line which touches the curve at P, PT is the direction in which the body is moving, and if gravity were suddenly to cease to operate upon the body it would continue to move ever afterwards uniformly in the direction PT. At each instant then the direction and the velocity of the body are being changed by the action of gravity, and the difficulty of the problem consists in determining the manner in which the effect of gravity in so changing the motion is to be estimated.

Now the method of resolving velocities, which was fully explained in Chap. I. will assist us in this difficulty. We have seen (Art. 10. p. 8) that to give the two resolved parts of the velocity of a body in given directions is the same thing as to give the actual velocity of the body and the direction of it's motion; suppose then that instead of speaking of the body P having a velocity in the direction PT, we say that

it has a vertical velocity u, and a horizontal velocity v; then if we can ascertain how u and v change from one instant to another, we shall know at any instant the actual velocity of the body, and the direction of its motion. Now the principle upon which we do this, and which we shall state more generally and explain presently, is as follows: we assume that the vertical velocity u will be affected by the vertical force of gravity, precisely as if the body were falling vertically, and that the horizontal velocity v will not be affected at all because there is no horizontal force. In other words, the horizontal distance of P from the starting-point A will increase uniformly, and the vertical height of P above A will change precisely according to the same law as if it had been thrown up vertically. If therefore a body be projected with a velocity V at an angle a with the horizon, so that the horizontal velocity is $V \cos a$, and the vertical velocity $V \sin a$, and if at the time t, h be the horizontal distance, and k the vertical distance (AN, PN in the figure) of the body from A; then

$$h = V \cos a \, . \, t,$$

$$\text{and } k = V \sin a \, . \, t - \frac{gt^2}{2},$$

and these two equations entirely determine the position of the body at the time t.

3. We shall not pursue this problem any further, because the whole subject of projectiles will come before us presently; we have discussed it here because this actual case seems to afford the easiest method of pointing out the general nature of the problem to be solved when a body is acted upon by forces not in the direction of its motion. The method of solution is precisely the same as that above adopted, namely, to regard the motion of the body with respect to two fixed directions, and to consider that the velocities in those directions are modified by the forces resolved in the same directions precisely as though the motion were rectilinear. The principle is gene-

rally otherwise stated, and is known as the *second Law of Motion*; it is as follows.

When any number of forces act upon a body in motion, each generates a velocity in its own direction precisely as it would if it acted singly upon the body at rest.

Stating the law thus we must suppose that at each instant each force generates a velocity in its own direction, and that all velocities so generated are compounded by the rule of the parallelogram of velocities. In all applications of the principle however we consider the motion of the body as referred to two fixed directions, and consider how these motions will be affected by the forces resolved in the same directions; and this appears to be the simplest mode of apprehending the law.

4. When we first introduced this principle, we said that we *assumed* that the vertical velocity of the projectile was affected only by the vertical force, and that the horizontal velocity was not affected because there was no horizontal force; and in truth the principle so stated seems scarcely to require proof and to be such as may be safely assumed; nevertheless, the second law of motion may be said to be capable of experimental verification. Examples in which a body is in motion and is acted upon by a force not in the direction of its motion are numerous; and the application of the second law of motion is easily seen. Thus if a vessel be sailing uniformly, and a ball be let fall from the top of the mast, it will fall at the foot of the same notwithstanding the motion of the ship; for when the ball is dropped, it has the horizontal velocity of the ship, and as the ship continues to move with this velocity, the ball falls relatively to the deck as though the ship were at rest. In reality the ball describes a curvilinear path, and at each moment its velocity and direction of motion are being changed by gravity; but gravity does not affect the horizontal motion, because it acts vertically, and therefore the ball falls at the foot of the mast precisely as though neither itself nor the ship had any horizontal velocity. In like

manner our motions on the deck of a ship moving uniformly, or on the floor of a railway carriage in motion, are quite independent of the motion of the ship or carriage; if we jump we descend upon the very spot from which we rose, and we can walk in any direction precisely as though we were standing upon a fixed platform. So likewise, when in exhibiting feats of horsemanship a rider standing upon his horse in rapid motion, jumps through a hoop and descends upon the horse again, it is only necessary for him to spring vertically upwards, the horizontal motion which he has already and which the horse retains being precisely the motion necessary to bring him again to the saddle. Numberless other instances might be adduced, in which gravity acts upon a body already in motion; examples of bodies in motion acted upon by a variety of forces are not so familiar, the heavenly bodies may however be referred to for illustration; thus, the motion of the moon is due not to the attraction of the earth only, but also to that of the sun, and the motion of each one of the planets is the result of the action upon it of the sun and of every other planet in the system. And we refer to this illustration because the successful application of our mechanical principles to the determination of the motion of the heavenly bodies is one of the best proofs of the correctness of the principles themselves; for in celestial mechanics we are free from disturbing causes, which affect the greater number of terrestrial problems, and the results of mathematical investigations can be tested by astronomical observations with a degree of exactness which unscientific persons can scarcely conceive. Now the motion of the heavenly bodies has been calculated, and the calculation proceeds upon the principles we have explained amongst others, and the result assures us (if we needed any assurance) that the second law of motion is correct in its principle.

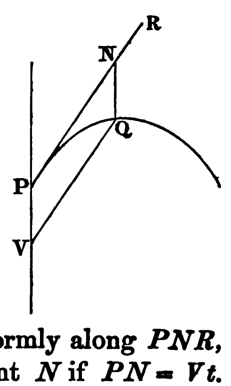
5. It will be observed that we have spoken of resolving the velocity of a body in motion into *two* fixed directions, and considering the changes of those velocities indepen-

dently; that is, we have spoken as if the motion were all in one plane; this, of course, is not necessarily the case; we shall however not be concerned with any problem in which it is not so, and we have therefore preferred speaking of that kind of motion in order that what was said might be more simple. The second Law of Motion does not, as will be at once perceived, imply any such restriction, and the principle of treating a problem is precisely the same, whether the motion be all in one plane, or not; practically, however, the difficulty is much increased.

It may also be observed that in this chapter we shall be able still to confine our attention to *accelerating force;* that is, we shall not be concerned with the mass of the body moved, but shall entirely confine our attention to the path described by a body, and the change of velocity in that path. In the case of a projectile, which will be the only problem of which we shall treat, the motion (supposed to take place in vacuum) will be precisely the same whether the particle which moves have the density of lead or of charcoal.

We now proceed to the determination of the path of a projectile, and the discussion of various cognate problems.

6. PROP. *The path of a projectile will be a parabola.*



Let the plane of the paper be the plane of the body's motion, that is, the vertical plane containing the straight line in the direction of which the body is projected, or the *direction of projection.*

Let P be the point from which the body is projected, and let it be projected in the direction PNR with a velocity V. Then PNR is a tangent to the curve in which the body moves; and if no force acted upon the body it would move uniformly along PNR, and at the time t would be at the point N if $PN = Vt$. Now from N draw NQ vertically downwards and equal to

$\frac{gt^2}{2}$; then NQ is the vertical space through which the body would fall in the time t in consequence of the action of gravity, and therefore according to the principles already explained, Q will be the actual place of the body at the time t.

Complete the parallelogram $PVQN$, then

$$PV = NQ = \frac{gt^2}{2},$$

$$\text{and } QV = PN = Vt;$$

$$\therefore\ QV^2 = V^2t^2 = \frac{2V^2}{g}.PV.$$

But in the parabola $QV^2 = 4SP.PV$, (Conics). Hence Q lies in a parabola, of which the axis is vertical, and the distance of P from the focus or from the directrix is $\frac{2V^2}{g}$. Q.E.D.

7. Prop. *To determine the velocity at any point of the path of a projectile.*

It appears from the preceding proposition, that if P be the point of projection, V the velocity of projection, and S the focus of the parabola, then

$$V^2 = 2g.SP.$$

Now any point in the path may be regarded as the point of projection, and the corresponding velocity as the velocity of projection, hence the preceding formula gives us the velocity at any point P of the path of a projectile, the path itself being supposed to be known.

If from P we draw a perpendicular upon the directrix of the parabola, and call this perpendicular PM, we have $SP = PM$, by the property of the parabola, and hence

$$V^2 = 2g.PM.$$

And it may therefore be stated that *the velocity at any point of the path of a projectile* is that which would be

acquired in falling from the directrix to the point in question. (See Art. 16. p. 33).

8. The velocity at any given time may also be found without supposing the path of the projectile to be given. Let v be the velocity at the time t, and let θ be the angle which the direction of the motion makes with the horizon; so that $v\cos\theta$ is the horizontal velocity, and $v\sin\theta$ the vertical velocity; then if V be the velocity of projection, and α the angle of projection, or the angle which PNR (see fig. p. 62) makes with the horizon, we have, according to our principles,

$$v\cos\theta = V\cos\alpha,$$

$$v\sin\theta = V\sin\alpha - gt;$$

$\therefore$ squaring the opposite sides of these equations and adding, we have

$$\begin{aligned} v^2 &= V^2\cos^2\alpha + (V\sin\alpha - gt)^2, \\ &= V^2 - 2Vgt\sin\alpha + g^2t^2, \\ &= V^2 - 2g\left(Vt\sin\alpha - \frac{gt^2}{2}\right). \end{aligned}$$

This formula gives us what we require, and it admits of an interpretation which it may be worth while to notice. $Vt\sin\alpha$ is the vertical height through which the body would have risen had gravity not acted, consequently $Vt\sin\alpha - \dfrac{gt^2}{2}$ is the actual vertical rise of the projectile above the point of projection during the time t; call this h, then we shall have

$$v^2 = V^2 - 2gh,$$

which is precisely the same formula as we should have had if the body had been projected vertically; and we may hence conclude that if from any point a number of balls be projected in different directions, but all with the same velocity, then the velocity of all will be the same when they cross any given horizontal plane, or when they are all at the same distance from the ground.

9. From the above equations we may also determine the direction of the body's motion at any given time, without reference to the properties of the parabola; for we have, by dividing one equation by the other,

$$\tan\theta = \frac{V\sin\alpha - gt}{V\cos\alpha} = \tan\alpha - \frac{g}{V\cos\alpha}t.$$

From this formula we perceive that $\tan\theta$ diminishes uniformly as t increases; when $\theta = 0$, the body is moving in a horizontal direction, it has, in fact, reached the highest point of its path, and is about to descend again; this will take place when

$$V\sin\alpha - gt = 0,$$

$$\text{or } t = \frac{V\sin\alpha}{g}.$$

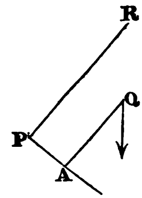
It may be noticed, that after having reached this highest point, the body describes in descending an exactly similar path to that which it described in ascending, and therefore the above value of t is half the time between the instant of the body leaving the ground and the instant of striking it upon its descent. We shall, for distinctness' sake, presently investigate the time of flight independently.

10. As one other illustration of the preceding formulæ, which it will be observed have no reference to the fact of the path being a parabola, we will determine at the time t the angle which the direction of the body's motion makes with the direction of projection, in other words how much the direction of motion has *deviated* from the original direction. This angle of deviation will be measured by $\alpha - \theta$, and we have

$$\tan(\alpha - \theta) = \frac{\tan\alpha - \tan\theta}{1 + \tan\alpha\tan\theta} = \frac{gt}{V\cos\alpha + \tan\alpha\,(V\sin\alpha - gt)}$$

$$= \frac{gt\cos\alpha}{V - gt\sin\alpha}.$$

This result may be obtained by a direct process, and it will illustrate our principles to do so. Let PR be the direction of projection as before, and instead of considering the motion with reference to a vertical and a horizontal line, let us consider it with reference to PR and a line PA at right angles to it; then the resolved part of g in the direction of PR is $g \sin \alpha$; and that perpendicular to it is $g \cos \alpha$; consequently at the time t,



$$\text{velocity parallel to } PR = V - gt \sin \alpha,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad PA = gt \cos \alpha;$$

and if ϕ be the angle which the direction of the motion makes with PA, we have, according to our principles,

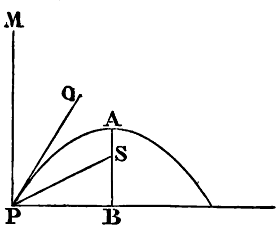
$$\tan \phi = \frac{V - gt \sin \alpha}{gt \cos \alpha},$$

and since ϕ is the complement of the angle which the direction of the motion makes with PR, that is, the complement of $\alpha - \theta$, this result coincides with that obtained before.

We now return to the consideration of the parabolic path of the projectile.

11. Prop. *To determine the position of the focus of the parabola described by a projectile, when the point of projection, and the velocity and direction of projection, are given.*

Let P be the point of projection, PQ the direction of projection; draw PM vertical, and at the point P in the straight line PQ make the angle QPS equal to the angle QPM; and make SP equal to $\dfrac{V^2}{2g}$, where V is the velocity of projection, S will be the focus required.



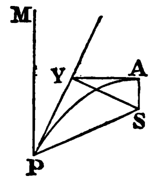
For since PQ is a tangent at P, and PM being vertical

is parallel to the axis, therefore the focus lies in the straight line SP which makes an angle with PQ equal to QPM. (Conics). Also, by Art. 7, $SP = \frac{V^2}{2g}$, whence the truth of the construction appears.

Cor. If the angle of projection be 45^0 the focus is in the horizontal line passing through the point of projection.

12. Prop. *To find the latus rectum of the parabola.*

From the focus S draw the perpendicular SY upon the tangent at the point P, that is, upon the direction of projection; let A be the vertex, then AY is a tangent at the vertex and therefore horizontal. (Conics).



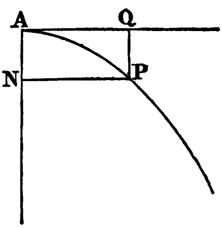
Now $SYA = 90^0 - a$, and $SPY = MPY = 90^0 - a$;

$\therefore$ the latus rectum $= 4AS = 4SY \cos a = 4SP \cos^2 a$

$$= 2\frac{V^2}{g}\cos^2 a.$$

Cor. If $a = 0$, the latus rectum $= \frac{2V^2}{g}$. This is the simplest case of the path of a projectile, and may perhaps with advantage be investigated independently.

13. Let A be the point of projection, which in this case will be the vertex of the parabola; AQ horizontal, QP vertical, AN vertical, PN perpendicular to AN. Then if P be the place of the body at the time t, we have



$$PQ = AN = \frac{gt^2}{2},$$

$$AQ = PN = Vt;$$

$$\therefore PN^2 = V^2t^2 = \frac{2V^2}{g} . AN.$$

Hence the path is a parabola, having for its latus rectum $\frac{2V^2}{g}$. (Conics).

14. PROP. *To determine the greatest height attained by a projectile.*

The greatest vertical height will be, according to the principles already explained, the height to which the body would rise if projected vertically with the vertical portion of its velocity of projection, that is, with the velocity $V \sin a$.

Hence the height required is $\frac{V^2 \sin^2 a}{2g}$.

COR. The height of the focus above the point of projection will therefore be $\frac{V^2 \sin^2 a}{2g} - \frac{V^2 \cos^2 a}{2g}$ (Art. 12), or $-\frac{V^2 \cos 2a}{2g}$. This has the appearance of being a negative quantity, but will not in reality be so, unless $2a$ be $< 90^0$, or $a < 45^0$. We have already seen that if $a = 45^0$, the focus is in the horizontal line passing through the point of projection, if $a > 45^0$ the focus is above that line, if $a < 45^0$ it is below the same.

15. PROP. *To find the horizontal range of the projectile, that is, the distance from the point of projection at which the path of the body meets the horizontal plane through the point of projection.*

The range will be equal to twice the distance of S from PM (fig. Art. 11), and therefore

$$= 2SP \sin 2a = \frac{V^2}{g} \sin 2a.$$

COR. The range will be greatest when $2a = 90^0$, or $a = 45^0$, that is, when the focus is in the horizontal line passing through the point of projection.

16. PROP. *To find the time of flight, that is, the time which elapses from the instant of projection until the instant*

of the projectile descending to the horizontal plane through the point of projection.

According to our principles, the time required will be that in which the horizontal range can be described uniformly with the body's uniform horizontal velocity. Therefore the time required

$$= \frac{V^2}{g} \sin 2a \div V \cos a = \frac{2V \sin a}{g}.$$

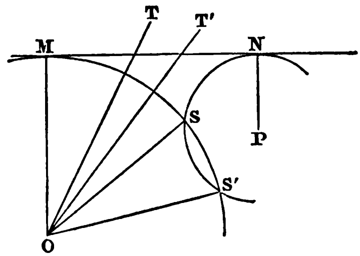
The same result may be obtained by consideration of the vertical motion of the projectile; for the time of flight will be the same as that required by a body projected vertically with the velocity $V \sin a$, to rise to its greatest height and return, that is, twice the time required by gravity to generate or destroy a velocity $V \sin a$, that is,

$$\frac{2V \sin a}{g}.$$

See also Art. 9.

17. Prop. *Given the velocity of projection, to construct for the direction so that the projectile may strike a given point.*

Let O be the point of projection; P the point through which the path of the projectile is to pass; V the velocity of projection.



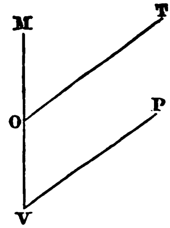
Draw OM vertical and equal to $\frac{V^2}{2g}$, and MN horizontal, which will be the directrix (Art. 7). Draw PN perpendicular to the directrix; and with centres O and P, at distances OM, PN, describe two circles cutting each other in S and S'. Then, by the nature of the parabola, either S or S' may be the focus of a parabola having MN for directrix, and passing through O and P; hence either S or S' may be the focus of the required path of the projectile.

Through O draw OT, OT', bisecting the angles MOS, MOS' respectively, then either OT or OT' may be taken as the direction of projection in order that the body may strike the given point P.

Cor. If the two circles do not intersect the problem is impossible, and if the circles touch there is only one direction of projection.

18. Prop. *Given the direction of projection, to construct for the velocity, in order that the body may strike a given point.*

Let O be the point of projection; P the point through which the path of the projectile is to pass; OT the given direction. Draw MOV vertical, and PV parallel to OT.



Take OM such that $PV^2 = 4OM \cdot OV$, that is, take OM a third proportional to OV and $\frac{PV}{2}$, then the horizontal line through M is the directrix, (Conics) and $V^2 = 2g \cdot OM$.

There is no ambiguity in this case, as in the preceding proposition; that is, there is only one velocity corresponding to a given direction of projection.

19. The preceding are some of the principal propositions relating to projectiles; problems and examples illustrative of them may be given to an indefinite extent, and we shall presently subjoin a few. Before leaving the subject, however, we will remark, that although it is true that a body acted upon by a uniform force which acts parallel to a given direction will describe a parabola, it does not follow, conversely, that if a body move in a parabola it is necessarily acted upon by a uniform force parallel to its axis. In order to make this conclusion hold we must have the condition, that the motion in the direction perpendicular to the axis is uniform; if this condition hold, we may then conclude that the motion parallel to the axis is *uniformly* accelerated, or that the body is acted upon by *a uniform* force in that direction.

20. We now conclude this chapter with some examples.

Ex. 1. A ball is projected with a velocity of 1000 feet per second, and strikes the ground at a distance of 100 yards; find the angle of projection.

Let α be the angle; then we have (Art. 15),

$$\frac{V^2}{g}\sin 2\alpha = 300,$$

$$\text{and } V = 1000,$$

$$\therefore \sin 2\alpha = \frac{300 \times 32.2}{1000^2} = \frac{96.6}{10000},$$

$$\therefore \log \sin 2\alpha = 7.9849771 \text{ by the tables,}$$

and $2\alpha = 33' 12''$ nearly, or the supplement of this angle;

and $\alpha = 16' 36''$ or the complement of the same.

Ex. 2. A ball fired from a gun at an elevation of 25° strikes the ground at a distance of four miles; find the velocity of the ball on leaving the gun.

If V be the velocity we have,

$$V^2 = \frac{32.2 \times 4 \times 1760 \times 3}{\sin 50^\circ},$$

$$\therefore \text{ by the tables, } \log V = 2.9741479,$$

$$\text{and } V = 942.21 \text{ feet per second.}$$

Ex. 3. Find the greatest range possible when the velocity of projection is 10 times that which would be acquired by a heavy body in falling during one second.

The greatest range is measured by $\frac{V^2}{g}$ (Art. 15. Cor.), and in this case $V = 10g$; hence the range required is $100g$ or 3220 feet.

Ex. 4. A body is projected with a velocity of 1000 feet per second and at an angle of 30°; how long a time will elapse before it strikes the ground?

In general the time of flight $= \frac{2V}{g}\sin\alpha$. (Art. 16)

$$= \frac{2000}{32.2} \times \tfrac{1}{2} \text{ in the present instance,}$$

$$= \frac{1000}{32.2} = 31.057 \text{ seconds.}$$

Ex. 5. In the preceding example, find the height of the body above the ground after the lapse of 25 seconds.

In general the height at the time $t = V \sin \alpha . t - \frac{gt^2}{2}$,

$$= 1000 \times 25 - 16.1 \times 25^2, \text{ in this case,}$$

$$= 25000 - 10062.5,$$

$$= 14937.5 \text{ feet.}$$

Ex. 6. Given the direction of projection and the height to which the velocity of projection is due, to construct for the greatest height to which the projectile will rise, and for the horizontal range.

Let P be the point of projection, PM the vertical height through which the body must fall in order to acquire the velocity of projection, PQ the direction of projection, PA horizontal.

From M draw MQ perpendicular to PQ, and from Q draw QR perpendicular to PM; then PR will be the greatest height required, and $4RQ$ the horizontal range. The student can supply a figure, and the proof of the construction.

Ex. 7. Or the range may be constructed as follows:

Draw MQ perpendicular to PQ as before; produce PQ, and take PF equal to $4PQ$; from F drop the perpendicular FH upon the horizontal line PH, PH will be the range.

Ex. 8. Given the range and the height to which the velocity of projection is due, to construct for the direction of projection.

In the construction of Ex. 6, it will be seen that a semicircle described upon PM as diameter will pass through Q, since MQP is a right angle.

Hence, conversely, if PH be the given range, and we make PN equal to one fourth of PH, and draw NqQ vertical, it will generally intersect the semicircle described upon the given line PM in two points Q *and* q; and PQ, Pq will be two directions of projection which will *satisfy the* conditions of the problem.

And it will appear that there will be only one direction of projection if the range equal twice the height to which the velocity is due, that there will be two directions if the range be less than this, and that the problem is impossible if the range be greater.

CONVERSATION UPON THE PRECEDING CHAPTER.

P. Is the theory of projectiles which we have been discussing nearly true experimentally?

T. Very far from it; the resistance of the air is so great, that the path of a cannon-ball differs extremely from what it would be in vacuum, and the determination of the path is extremely difficult in a mathematical point of view, indeed the problem cannot be completely solved.

P. The propositions which I have been reading must therefore be regarded as purely theoretical?

T. They are of little practical value in the science of gunnery, but they are of great use as illustrations of mechanical principles, and that is the reason why they have been so fully discussed. Nothing can be found better as an illustration of the second law of motion than the case of a projectile, because it is that of a body acted upon by a single uniform force not in the direction of the motion. The most general problem we can have is that of a body acted upon by any number of forces in any directions; now in the case of the projectile we have only *one* force, and that force always acting in the same direction. Hence, the motion of a projectile may be regarded as next in simplicity to that of a body falling vertically.

P. The determination of the motion appears in fact to be reduced by the method given in Art. 2, to that of a falling body.

T. Yes; that is the result of considering the motion of the body as compounded of two, and regarding each of

these two motions as a rectilinear motion. And according to the principles laid down in this Chapter, each of these supposed motions is affected by the force in that direction, exactly as though the body were moving in a straight line under the action of the force. This is, I think, the simplest mode of apprehending the second law of motion.

The clear apprehension of the law, as well as the parabolic theory of projectiles which immediately follows from it, is due to Galileo; the fact is, that the parabolic form of the path is so thoroughly destroyed by the resistance of the air, that the notion which was at one time prevalent of the path being in the first instance a right line, and then becoming a curve and finally becoming a straight line again, does in reality much more nearly correspond to the actual path than does the parabolic theory. Nicholas Tartaglia (who flourished in the early part of the sixteenth century) was the first who perceived that the path must in reality be curvilinear throughout, but he did not succeed in determining the nature of the curve. This was effected by Galileo, and his treatise on the subject forms the substance of his fourth dialogue, which is entitled *De Motu Projectorum.* I entered at some length into an account of his dialogue on falling bodies, because it is interesting to see how these subjects presented themselves to the minds of early discoverers, and for the same reason I will call your attention (but only briefly) to the dialogue on projectiles.

The dialogue is introduced as follows:

Quæ in Motu æquabili contingunt accidentia itemque in Motu naturaliter accelerato super quascunque planorum inclinationes, supra consideravimus. In hac, quam modo aggredior, contemplatione, præcipua quædam symptomata, eaque scitu digna in medium afferre curabor, eademque firmis demonstrationibus stabilire, quæ Mobili accidunt dum motu duplici latione composito, æquabili nempe, et naturaliter accelerata, movetur: hujusmodi autem videtur esse Motus ille, quem de Projectis dicimus: cujus generationem tale constituo.

Mobile quoddam super planum horizontale projectum mente concipio omni secluso impedimento: jam constat ex his quæ fusius alibi dicta sunt illius motum æquabilem, et perpetuum super ipso plano futurum esse, si planum in infinitum extendatur: si vero terminatum, et in sublimi positum intelligamus, mobile, quod gravitate præditum concipio, ad plani terminum delatum, ulterius progrediens, æquabili, atque indelebili priori lationi superaddet illam, quam a propria gravitate habet deorsum propensionem, indeque motus quidam emerget compositus ex æquabili horizontali, et ex deorsum naturaliter accelerato: quem Projectionem voco. Cujus accidentia nonnulla demonstrabimus.

I may remark that nothing can be more distinct than the explanation of the composition of motion here given: it is, in fact, the enunciation of the second Law of Motion, upon which the motion of projectiles immediately depends. The discourse contains fourteen propositions, which begin with the fundamental one of the motion of the projectile being parabolical, and then proceeds to various applications, much in the same manner as in a modern treatise on the subject.

THEOR. PROP. I. Projectum dum fertur motu composito ex horizontali æqualibi, et ex naturaliter accelerato deorsum, lineam semiparabolicam describit in sua latione.

This is demonstrated in the same manner substantially as in the preceding Chapter.

In the conversation which follows this proposition, the question of the apparent disagreement of the parabolical form with experiment is discussed, and referred to its true cause, the resistance of the air; and in the course of it Salviati explains that if a body be let fall in a medium such as the air, the acceleration of its motion will not increase continually, but that the resistance increasing as the velocity increases will at length destroy the acceleration, and that the body will thus continually approximate in its motion to a uniform terminal velocity; a result which mathematical investigation confirms.

The next proposition does not distinctly refer to the theory of projectiles; after it Galileo introduces a variety of Theorems and Problems, which are of no special interest except as shewing how thoroughly he had mastered the principles of the motion.

P. I am surprised to find that the resistance of the air is so great as to render the parabolic theory of projectiles practically useless.

T. The resistance of the air depends upon the velocity with which the body passes through it; it is usual to consider the resistance to vary as the square of the velocity, a law which though not strictly true is sufficiently near to the truth to give some notion of the enormous effect of the air for high velocities, and you will remember that in all cases of practical importance the velocity is very great. Thus in some cases it has been found that the resistance of the air was equivalent to as much as 100 times the force of gravity; this was the case when the velocity was about 2000 feet per second or 23 miles in a minute. Now it is evident that such a force as this must totally destroy the results obtained upon the supposition of the ball moving in a vacuum.

And without entering into any consideration of the form of the curve, the impossibility of obtaining anything like true results upon the parabolic theory is manifest from this, that with a velocity such as has been mentioned and an angle of projection equal to 45^0 the range ought to be almost 24 miles, (Art. 15); whereas in practice the range is frequently found to be short of one mile.

P. It is clear from this that the parabolic theory must remain theory only. I suppose that the mode of determining the motion of the projectile given in Art. 2, in which no reference is made to the parabola, is a complete solution of the problem of motion in a vacuum.

T. Yes, the velocity and position of the body at any given time can be assigned; and when that can be done the problem is completely solved; and indeed it would be

possible from the equations there given to shew that the body moved in a parabola, but to do so would require another method of treating the parabola from that with which you are acquainted. You will observe that in determining the height to which the body rises, (Art. 14) and the time of its flight, (Art. 16), the fact of the parabolic form of the path is not introduced, but reference is immediately made to the principles of the Article to which you have referred.

P. The difference between the two methods is in fact nothing more than taking different straight lines to which to refer the motion.

T. It is merely that; and it depends upon circumstances what lines of reference may happen to be most convenient. Suppose, for instance, we project a body from the foot of an inclined plane, and we wish to know the greatest distance from the plane attained by the body, then we need only consider the motion in a direction perpendicular to the plane.

Let θ be the inclination of the plane, and a the angle of projection; then the velocity perpendicular to the plane will be $V \sin(a - \theta)$, and the resolved part of gravity perpendicular to the plane will be $g \cos\theta$; you will see the truth of this if you draw a figure; hence we have at once for the greatest distance attained by the projectile

$$\frac{V^2 \sin^2(a - \theta)}{2g \cos\theta}.$$

In like manner if we call x the distance of the body from the plane at the time t, we shall have

$$x = V \sin(a - \theta)\, t - \frac{g \cos\theta}{2} t^2;$$

and therefore the time which elapses before the body strikes the plane will be $\dfrac{2V \sin(a - \theta)}{g \cos\theta}$. You may upon the same principles find the *range* upon the inclined plane, that is, the distance from the point of projection at which

the body strikes the plane. This I leave to your own ingenuity.

P. Are there no other problems besides that of projectiles to which the principles of this chapter apply?

T. The *principles* apply to the very important problem of the motion of the heavenly bodies; for instance, the earth and each of the planets describes an ellipse about the sun in one of the foci, and it is possible to shew that this elliptical form of the orbit is an immediate consequence from the hypothesis of the sun attracting the planets according to a certain law, and no other principle is required for the demonstration of this remarkable proposition besides those of the first and second Laws of Motion; but the force not being a uniform force, we should find considerable difficulty in applying our principle, and had better therefore defer the problem.

EXAMINATION UPON CHAPTER III.

1. Enunciate and illustrate the second law of motion.

2. Apply it to find the height above the ground at the time t of a body, which is projected with a velocity V and in a direction making an angle α with the horizon.

3. A ball is thrown with a velocity $6g$ at an angle of 60° with the horizon, find after what lapse of time it will strike a wall at a distance of 10 feet.

4. The path of a projectile is a parabola.

5. Shew that the velocity at any point of the parabolic path of a projectile is that which would be acquired in falling from the directrix.

6. A body is projected with a velocity of 100 feet per second; find the position of the directrix.

7. Determine the greatest height to which the projectile will rise, and the range on a horizontal plane.

8. Find the time of flight.

9. *If* the angle of projection be 45°, the focus lies in the horizontal *plane* passing through the point of projection.

10. Determine the angle of projection for which the range is a maximum.

11. Given the velocity, determine the angle of projection in order that the projectile may pass through a given point.

12. Given the range, determine the angle in order that the same result may be produced.

13. Prove that the two angles given by the construction in the preceding question are complementary to each other.

14. Given the angle of projection, determine the velocity in order that the projectile may pass through a given point.

15. The curve traced out upon an inclined plane by a body projected along its surface will be a parabola.

16. A body is projected from the foot of an inclined plane; find the distance from the foot at which the body strikes the plane; the velocity and angle of projection and the inclination of the plane being given.

17. Find also the time which elapses before it strikes the plane.

18. A body is projected horizontally from the highest point of a vertical circle; shew that in order that the body may *clear* the circle, the velocity of projection must not be less than $\sqrt{gr}$, where r is the radius of the circle.

19. If the velocity be precisely $\sqrt{gr}$ in the preceding problem, find the distance of the focus from the centre of the circle.

20. Find also under the same circumstances the distance of the projectile from the centre of the circle, when it passes the horizontal plane in which the centre lies.

21. From the vertex of an equilateral triangle having its base horizontal and plane vertical, a body is projected horizontally in such a manner as to pass through one extremity of the base; compare the perimeter of the triangle with the height due to the velocity of projection.

22. Find the distance of the focus from the vertex of the triangle in the preceding problem.

23. A body is projected upwards upon an inclined plane, l feet long, inclination α, with a velocity V; find the latus rectum of the parabola *described after* the body leaves the plane.

24. A ball is projected with velocity of V at an angle of $60°$ with the horizon; find how long a time will elapse before it is moving in a direction making an angle of 45^0 with the same.

25. Given the range and the time of flight, find the velocity and angle of projection.

26. Give the result of the preceding question, when the time of flight in one second, and the range 16.1 feet.

27. The time is noted at which a projectile is moving at an angle of 45^0 with the horizon, and again when it is moving horizontally, the time being measured from the instant of projection; with these data find the velocity and angle of projection.

28. Shew that if the distance of a body from the point of projection at any two given periods of time measured from the instant of projection be given, the velocity and angle of projection can be found.

29. If the resistance of the air were imperceptible, shew that the accelerating force of gravity might be determined from the motion of a cannon-ball, without determining the velocity with which the ball issued from the gun. What observations would be necessary?

30. A shell is thrown at an angle α, and falls a feet short of the mark intended; another is thrown with precisely the same charge of powder, but at an angle of elevation greater by a given quantity β, and the shell now falls b feet beyond the point; supposing the velocity of projection of the shell to be known, shew how to find the angle of elevation in order that the shell may fall at the point desired; and prove that if $a = 1$ foot, and $b = 1$ foot, the angle of elevation will be $\alpha + \frac{\beta}{2}$ early.

CHAPTER IV.

ON THE MOTION OF BODIES UNDER THE ACTION OF FORCE, WHEN THE MASS OF THE BODY MOVED IS TAKEN INTO ACCOUNT. THIRD LAW OF MOTION. FALLING BODIES CONNECTED BY A STRING.

1. It will be remembered, that in speaking of the action of force as changing the motion of a body, we discriminated between the case in which we are able to confine our attention entirely to the acceleration or retardation of the motion and the case in which we have to take into account not only the change of velocity but also the mass of, or quantity of matter contained in, the body moved. In the former case we called the force *accelerating* force, in the latter *moving;* but we took especial pains to point out that these terms did not imply two different kinds of force, but only two different modes of estimating it. Moving force we have hitherto not discussed, because we found that there was an important class of problems, namely, that of falling bodies and projectiles, which we could treat while we regarded force purely in its accelerating or retarding character, and we preferred to treat of force considered thus, before we took the more general view of it as moving force.

2. In the present Chapter we propose to treat of moving force, or to consider the action of force in those cases in which it is necessary to take account of the mass of the body moved. And here the question immediately arises, what do we mean by the *mass* of a body? It may be answered, that we mean the quantity of matter it contains; but this answer only shifts the difficulty, because we are compelled to ask, how do we know anything concerning the quantity of matter which a body contains? the

mere magnitude of a body evidently does not give us the information we require, for a body may be compressed or expanded without in any way altering the quantity of matter contained; we require then a definition of mass, or else some distinct method of measuring the quantity of matter.

3. Now the question, how is the mass of a body to be measured, assumes a very different appearance, according as we regard it as a *statical* or as a *dynamical* question; and it will assist the student in understanding the subject to regard it in both ways.

4. First let us regard the question merely as a statical one, that is, let us consider how we shall estimate the mass of a body without any reference to the manner in which difference of mass may modify the effect of a force in changing a body's velocity. Now if we take a body of a certain magnitude and compress it to half that magnitude, it is clear that we do not alter the quantity of matter in it; the change produced we express by saying, that we have reduced it to half the original *volume* and have doubled its *density*, and that the *mass* consequently remains the same. If we had *weighed* the body before we compressed it, and if we *weigh* it again after we have compressed it, we shall find that the weight is the same; and this suggests to us that weight may be a proper test of mass or of the quantity of matter; and it will give us a simple and distinct notion of mass if we say that the mass of two bodies is the same when the weight is the same, and if the word mass be not otherwise defined, we may of course, if we please, thus define it. When however we say that the weight of a body measures the *quantity of matter* in the body, we do to a certain extent advance a theory concerning the earth's attraction; for if we take a cubic inch of cork and a cubic inch of lead we find that the latter weighs much more than the former, and accordingly we say that there is much more matter contained in it; and by thus speaking we imply, that the cubic inch of

lead does not weigh more than the cork because it is lead, but because there is more matter in it; that is, we assume that the earth acts upon all kinds of matter with equal intensity, and that the difference between its action upon two bodies of the same volume is a difference depending upon quantity only and not upon quality. This, it will be remembered, is not the case with all kinds of attraction; the magnet acts upon iron and not upon wood, its action depends upon the quality of the matter submitted to it; and it would be difficult to assert *à priori* that the same was not the case with the attraction of the earth, nor indeed can it be easily proved without dynamical considerations that it is not so: we shall however assume the contrary, that is, we shall assume that the earth acts with equal intensity upon matter of all kinds, and upon this assumption the weight of a body will measure its mass or the quantity of matter it contains. In fact, in Statics we can assign no other definition of mass than by saying, that *two bodies have the same mass when they have the same weight.*

5. Before we leave this part of the subject, we will put down two or three equations which embody the meaning of the preceding paragraph. Let W be the weight of a body, M its mass, V its volume or the number of cubic inches it contains, D its density. Then all that we can say of M is that

$$M \propto W,$$

or if M', W' be the mass and weight of another body,

$$M : M' :: W : W'.$$

M, V, and D are connected by the equation

$$M = DV,$$

and this gives the only definition of D. The equation expresses what we have already stated, namely, that if we take a body of given mass and compress it to half its volume we double its density, in other words, if the mass of a body be given, the density varies inversely as the volume.

6. Now let us look upon the question of the method of measuring the mass of a body from a dynamical point of view. And for fear of any misunderstanding let it be first remarked, that if a body be at rest any force however small will put it in motion; practically there are always impediments to motion such as friction and the like, and a body will not move from rest unless the force acting upon it be sufficient to overcome such impediments; but if there be no such hinderances to motion the body itself offers no resistance; it is true that if a body were placed under such circumstances, we should be sensible of an effort in putting it in motion, but this sensation would not imply a resistance on the part of the body; so far as extraneous forces are concerned matter is *inert*, it remains at rest until a force acts upon it, and then, however small the force may be, it acquires motion.

Suppose that we have the means of exerting a given uniform force by means of a spring or by any other method. And suppose that with this force we make experiment upon different bodies, and that we have the means of observing accurately the velocity generated in each in the period of one second. Let there be two bodies, which we will denote for clearness' sake by M and M', and in one second let this given force generate the same amount of velocity in M' as in M; in fact, having first made experiment upon M, and found what velocity can be generated in one second, find another body M' in which the velocity generated is the same; then we may, if we please, describe the relation between M and M' by saying that they have the same mass; in doing so we are putting aside for the moment the statical notion of mass and adopting a new notion, namely, that *two bodies have the same mass, or contain the same quantity of matter, when the same force generates in them the same velocity in the same time.* Now let us fasten M and M' together, then it will be found that in the compound body the force employed in the former experiments will generate only half the velocity in one second; and generally it will appear that the velocity

generated in one second will be inversely proportional to the mass of the body upon which the force acts, mass being defined as in the present Article. Suppose therefore that we denote force by F, the mass of the body by M, and the velocity generated in a second by f, then when F is given, it appears from what precedes, that

$$f \propto \frac{1}{M};$$

and when M is given we know that

$$f \propto F,$$

because F is measured in that case by the velocity generated in one second, as explained in a former Chapter; hence, when neither M nor F are given, we have by the general principle of variation,

$$f \propto \frac{F}{M},$$

$$\text{or } F \propto Mf.$$

In other words the *moving force* F varies jointly as the *mass moved* M, and the *accelerating force* f.

7. We have now treated of the mass of a body statically, and we have also treated of it dynamically; or rather we have given a statical measure of something which we have called *mass*, and we have given a dynamical measure of something which we have also called *mass*, but we have yet to see whether the two things independently defined and both designated by the name *mass* are the same; the student may suppose, from the fact of our assigning the same name in the two cases, that this is so, but he must remember that we have not proved it. This is what we now proceed to do.

From the formula $F \propto Mf$ we perceive, that if in two cases the accelerating force (f) be the same, the mass M (dynamically defined) will be proportional to the moving force F. Now treating the question statically, we assumed that mass was measured by weight, that is, by the whole

moving force of the earth's attraction upon it, and it will therefore follow that the two modes of estimating mass come to the same thing, *provided it be shewn that the accelerating force of the earth's attraction upon all bodies is the same.* This is in fact the case, as has been before mentioned (Art 6. p 24); if bodies be allowed to fall under such circumstances that the air cannot act upon them, all bodies fall with velocities equally accelerated; whether a body be composed of iron, wood, feathers or what not, a velocity of 32.2 feet is generated in one second: in other words f is the same for all bodies when the force acting upon them is the earth's attraction or their own weight: consequently $M \propto F$, or the mass varies as the weight, according to our statical definition.

8. It comes to the same thing then whether we say that two bodies are of the same mass when they have the same weight, or that two bodies are of the same mass when the same force acting upon them generates in them equal velocities in equal times. The former is the simpler definition and the one generally referred to when we speak of mass. Suppose that we so define mass, and let us denote the product of the mass of a body and its velocity at any time by the term *momentum;* that is, if M be the mass, and V the velocity, let us call MV the *momentum;* then the formula $F \propto Mf$, expresses that *the momentum generated in one second is proportional to the moving force.*

9. And thus we are led to the discovery of the proper measure of moving force, that is, the proper measure of the effect of a pressure or tension upon a body when the mass of the body is taken into account. Moving force may be either uniform or variable; if uniform, it is measured by the *momentum* generated in a unit of time, or one second; if variable, by the momentum which would be generated in a unit of time, on the hypothesis of the force remaining uniform during that unit.

If then we define *mass* as being proportional to *weight,* and *momentum* as being the product of the *mass* and the

velocity, we may enunciate the following as the *Third Law of Motion.*

When a uniform force acts upon a body, the momentum generated in a unit of time is proportional to the force; and when a variable force acts, the momentum which would be generated in a unit of time on the hypothesis of the force remaining uniform during the unit of time is proportional to the force.

In other words, we may take the measure of a moving force as being the product of the mass of the body and the accelerating force; or if F be a force, M the mass of the body, f the accelerating force, then we may say that

$$F = Mf.$$

We have already, it will be remembered, shewn that $F \propto Mf$, when F is uniform; the differences between the equation just now written down, and that which we had before are these, that at present we are not confining our attention to uniform force, and that we have used the symbol $=$ instead of $\propto$; this latter change merely implies, that the unit of moving force is that force which will produce a unit of momentum in a unit of time.

10. Let us illustrate the above equation by discussing it in the important case of the earth's attraction; in that case the force F is the *weight*, and in conformity with the practice adopted in statics we will therefore denote it by W; the accelerating force f is the velocity generated in a falling body in one second, or 32.2, which we have already agreed to denote by g: hence our equation becomes

$$W = Mg,$$

Mg therefore is the dynamical measure of the weight of a body; and this, instead of the symbol W, will be adopted as the symbol of weight in all dynamical problems.

11. Weight then, which in statics is measured by the number of pounds to which the weight of the body is

equivalent, is estimated in dynamics by the momentum which the body acquires in falling during one second: and the third law of motion as above enunciated asserts that these two measures are equivalent, that is, that if two weights appear to be one twice as great as the other according to the statical mode of comparing them they will appear to be in the same ratio if compared dynamically. It is manifest that the statical mode of comparing weights would not answer our purpose in the present subject, because it tells us nothing of the actual effect of the force of gravity in producing or changing a body's motion; and it is equally clear that it was unnecessary in statics to adopt any more complicated mode of measuring force, than that of considering what number of pounds weight would be sufficient to produce equilibrium, because no motion being actually produced it was not necessary to complicate the subject by considering what motion *might* be produced.

12. On the whole, the student must fix his mind as much as possible upon the notion of *momentum generated or destroyed being the measure of the action of force.* This may be well illustrated by reference to the case of one body striking another; in alluding to this, we anticipate to a certain extent what is to be contained in a subsequent Chapter, nevertheless the problem will be intelligible so far as we introduce it here. Suppose then a ball to be moving with a certain velocity, and suppose it to strike another ball which is at rest; then the striking ball loses *momentum* and the ball struck receives *momentum*, and the momentum lost by one is precisely equal to that gained by the other; and this is the case under all circumstances however complicated; a body can never lose any of its momentum without imparting to some other body or bodies precisely the amount of momentum which it itself loses.

The phrase *quantity of motion* is sometimes used instead of *momentum;* the two terms are exactly equivalent; and we may therefore say that a body cannot lose any

portion of its *motion* without communicating to some other body or bodies the same *quantity of motion* as it loses. The student will bear in mind, that in case the *motion* of a body should be thus spoken of, the term must be interpreted strictly in its scientific sense.

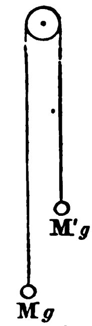
13. It may be worth while to remark, as illustrative of the relation between accelerating and moving force which we have been discussing, that accelerating force may be regarded as the moving force upon a unit of mass. For we have the general relation between the moving force F and the corresponding accelerating force f,

$$F = Mf;$$

now if we make $M = 1$, we have $f = F$; in other words, the accelerating force and the moving force to which it corresponds may be regarded as the same provided the mass moved be taken as unity.

14. The whole of the preceding discussion will doubtless appear difficult to the student, but it is hardly possible to present the subject under a form free from difficulties. Indeed it may be asserted that nothing but mature reflection can give clear views upon the fundamental principles of Dynamics. The subject in its present form is the result of the efforts of the most acute minds, and our present knowledge was only attained after many attempts and the explosion of many errors. It cannot be expected therefore that the subject should be otherwise than difficult; the application of our principles will however tend to make them intelligible, and to such application we now proceed, observing that we cannot carry the student beyond a very limited range without making use of mathematical methods with which we do not consider him to be acquainted.

15. Prop. *Two heavy bodies are connected by a fine string, which passes over a small pully; to determine the motion.*



Let M, M' be the masses of the bodies, M being greater than M'. Then the difference between the two weights will be the moving force which makes the heavier descend. We shall suppose that no part of this force is expended in turning the pully; practically this will not be the case, but we shall suppose it to be so, or we may consider the string to pass over a perfectly smooth horizontal cylinder, which will come to the same thing. The difference between the two weights Mg, $M'g$, will therefore be employed wholly in moving the weight; in other words, we shall have a moving force $Mg - M'g$, and $M + M'$ for the mass moved.

Hence by the principles which we have expounded, the accelerating force, or the velocity generated in one second, will be

$$\frac{M - M'}{M + M'} g.$$

And M will descend precisely as if it were free, and gravity reduced in the ratio of $M + M' : M - M'$. That is, the velocity of M downwards, or of M' upwards, at the time t, will be $\frac{M - M'}{M + M'} gt$; and the space described, will be

$$\frac{M - M'}{M + M'} \frac{gt^2}{2}.$$

Suppose, for instance, one of the weights to be twice as great as the other, or let $M = 2M'$; then the accelerating force will be $\frac{g}{3}$; in other words, the system will acquire a velocity of about 10.72 feet in one second, and in the first second each body will move through about 5.36 feet.

We have supposed the string connecting the bodies to have no weight; this we have done for the sake of simplicity; the only effect of the string which we have

considered is that of connecting the two bodies. In maintaining this connexion, the string must itself undergo a certain *tension*, that is, there must be a certain force necessary to prevent the bodies from separating the one from the other, and this force the string produces. Let us investigate the magnitude of the force, or the tension of the string.

16. Prop. *To find the tension of the string.*

The tension of the string will be measured by the momentum which it destroys in M, or which it generates in M', in one second.

If M fell freely, the momentum generated in one second would be Mg; as it is, the momentum generated is, as we have seen, $M\dfrac{M-M'}{M+M'}g$. Therefore the momentum destroyed in one second is

$$Mg - M\frac{M-M'}{M+M'}g, \text{ or } \frac{2MM'}{M+M'}g.$$

In like manner the momentum generated in M' in one second by the string is

$$M'g + M'\frac{M-M'}{M+M'}g, \text{ or } \frac{2MM'}{M+M'}g;$$

which is the same expression as before, as it ought to be, according to the general principle which we before explained, namely, that a body cannot lose *motion* without generating in another as much as it loses. The expression $\dfrac{2MM'}{M+M'}g$ therefore measures the tension of the string.

For instance, suppose that $M = 2M'$, then the tension is $\dfrac{2}{3}Mg$, or two-thirds of the larger weight; and we may if we please estimate the tension statically, thus if the weights be 6lbs. and 3lbs. the tension will be 4lbs., or the string will experience precisely the same tension as it it were

suspended by one extremity and had a weight of 4lbs. attached to the other.

17. This problem so well illustrates the principles which we have been endeavouring to explain, that we will solve it again, and in a slightly different manner.

Let us call the tension of the string T, so that $Mg - T$ is the moving force upon M, and $M'g - T$ the moving force upon M'. Then by our principles, $g - \frac{T}{M}$ is the velocity generated in one second in M, and $g - \frac{T}{M'}$ is the velocity generated in M'.

Now since the bodies are connected by an inextensible string, they must always move with the same velocity; but one must ascend and the other descend, or algebraically speaking, the velocity generated in one must be equal to that generated in the other but must have the opposite sign; hence

$$g - \frac{T}{M} = -\left\{g - \frac{T}{M'}\right\},$$

$$\text{or } T\left\{\frac{1}{M} + \frac{1}{M'}\right\} = 2g,$$

$$T = \frac{2MM'}{M + M'}g.$$

This is the same result as before; it has been obtained, as will be observed, without a previous investigation of the velocity generated in one second, in other words, of the accelerating force, and from it we can at once obtain the accelerating force, which according to the former method was investigated first.

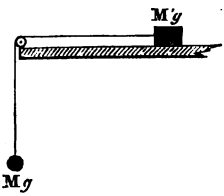
For we have seen that the velocity generated in M in one second is $g - \frac{T}{M}$; but

$$g - \frac{T}{M} = g - \frac{2M'}{M + M'}g = \frac{M - M'}{M + M'}g,$$

the same expression for the accelerating force as was obtained before.

18. If we had considered the pully over which the string passes as having a finite mass, instead of being indefinitely small, a portion of the moving force of the system would have been expended in turning the pully, and the tension of the two portions of the string would not have been the same; the problem thus varied cannot be solved by the principles already explained.

19. Let us vary the problem by supposing one of the bodies to rest upon a smooth horizontal plane, and the other to descend; the two being connected by a fine string passing over an indefinitely small pully as in the figure.



In this case, if MM' be the masses as before, the moving force will be Mg, and the mass moved $M + M'$; hence the accelerating force is $\dfrac{M}{M + M'} g$, and the problem is completely solved.

It appears from the comparison of this result with what is said in Art. 26, p. 40, that the motion of M' is precisely similar to that of a body upon a plane inclined to the horizon at an angle a, where $\sin a = \dfrac{M}{M + M'}$.

What will be the tension of the string in this case? It will be measured by Mg – the portion of the weight Mg which is employed in moving itself, *i. e.* by

$$Mg - M\frac{M}{M + M'}g \text{ or } \frac{MM'}{M + M'}g.$$

The tension then will be only half what it was in the preceding problem.

20. Let us obtain this result otherwise. Call the tension T; then

moving force upon $M = Mg - T$,

$$\therefore \text{ accelerating force} = g - \frac{T}{M};$$

$$\text{moving force upon } M' = T,$$

$$\therefore \text{ accelerating force} = \frac{T}{M'};$$

but the bodies move with equal velocities, or the accelerating forces upon them are the same;

$$\therefore g - \frac{T}{M} = \frac{T}{M'},$$

$$\therefore T = \frac{g}{\frac{1}{M} + \frac{1}{M'}} = \frac{MM'}{M + M'} g, \text{ as before.}$$

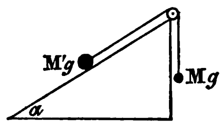
21. In Art. 26 p. 40, we observed, that in the case of a heavy body falling down a smooth inclined plane, the resolved part of the force of gravity in the direction of the plane accelerates the motion, and that the resolved part perpendicular to the plane causes pressure upon the plane, and we then confined our attention entirely to the former; we are now in a condition to treat the problem more completely.

Let a be the angle of inclination of the plane, Mg the weight of the body; then the two resolved parts of Mg will be $Mg \sin a$ parallel to the plane, and $Mg \cos a$ perpendicular to it. The accelerating force corresponding to the former will therefore be $g \sin a$, as in the Article before referred to; there is no acceleration perpendicular to the plane, consequently the whole of the moving force $Mg \cos a$ is expended in producing pressure upon the plane, in fact the effect of the plane is to destroy a velocity $g \cos a$ per second in the body. The pressure upon the plane will be precisely the same as in statics, for there being no motion perpendicular to the plane the question will be entirely statical so far as the forces in that direction are concerned.

This problem is worthy of the student's attention as presenting us a case in which one portion of the moving

force is entirely taken up in producing motion, the other in producing pressure. Force is in general expended in producing both, thus in the case of the falling bodies connected by a string, the moving force upon M partly moves M and partly produces tension in the string, and these parts we have already separately determined, but they are not so easily separated as in the simple case of the smooth inclined plane.

22. We may combine the two problems treated of in Arts 15 and 21; that is, we may suppose two bodies to be connected by a fine string and one of them to move upon a smooth inclined plane.



Let M hang freely and M' rest upon a plane whose inclination is α. Then the mass moved is $M + M'$, and the moving force is $Mg - M'g \sin \alpha$, supposing M greater than $M' \sin \alpha$ so that M shall descend. Hence the accelerating force is $\dfrac{M - M' \sin \alpha}{M + M'} g$.

23. Still more generally we may suppose each of the weights to rest upon an inclined plane; then if β be the inclination of that upon which M rests, the accelerating force will be $\dfrac{M \sin \beta - M' \sin \alpha}{M + M'} g$.

Let us determine the tension of the string in this case; call it T, then by our principle,

$$T = Mg \sin \beta - M \frac{M \sin \beta - M' \sin \alpha}{M + M'} g,$$

$$= \frac{MM'}{M + M'} (\sin \beta + \sin \alpha) g.$$

Suppose that $\beta = 90^0 - \alpha$, that is, let the two planes be at right angles to each other, then

$$T = \frac{MM'}{M + M'} (\sin \alpha + \cos \alpha) g,$$

$$= \frac{MM'}{M + M'} \sin (\alpha + 45^0) g \sqrt{2}.$$

This will be greatest when $a + 45^\circ = 90^\circ$, or $a = 45^\circ$; hence if two weights rest in the manner above supposed upon two planes at right angles to each other the tension of the string is greatest when each of the planes is inclined at an angle of 45° to the horizon.

CONVERSATION UPON THE PRECEDING CHAPTER.

P. This Chapter seems to me to present considerable difficulties; it is not so much that any one particular portion of it is hard to be understood, but I feel that I do not at present comprehend the whole scope of it.

T. I imagine that no one ever studied Dynamics without having the same feeling on occasion of his first acquaintance with this part of the subject; and probably you will find that only long acquaintance will give you an entire insight. But you will see the necessity of some such explanations and discussions as this Chapter contains by proposing to yourself a problem, and considering how you would solve it without the rules which this Chapter supplies. Take, for example, the problem which we have already solved; two weights of 1lb. and 2lbs. respectively, are connected by a fine thread which passes over a very small pully, or a smooth horizontal cylinder, what tension will the string experience? how would you solve this problem?

P. I confess that I should be quite at a loss.

T. Very well; then you see at least the need of some principle beyond those which enabled us to solve the problem of falling bodies and that of projectiles.

Let us discuss the problem now proposed, as we may suppose that we should have done if we had not known what the preceding Chapter has told us; we should perceive that in consequence of being connected by a string to another weight the 2lbs. weight would not move so rapidly as it would if free; the 1lb. weight on the other hand, so far from obeying the ordinary rule of a falling body, actually

moves in opposition to gravity, and ascends instead of descending; some of the motion of the 2lb. weight then is destroyed, or lost, in consequence of its connexion with the string. This destruction of motion is what causes the tension of the string, and the same tension not only destroys all the motion which the 1lb. weight would have had, but generates a motion opposite to that which it naturally would have.

Hence, I think we may say that the motion which is destroyed in one weight is transferred by the string to the other, and the communication of this motion from one to the other is that which causes the tension.

P. But what do you mean by *motion?* it seems rather a vague term.

T. That is precisely the point. In order that what I have now been saying to you should have any scientific value, it is necessary that we should be able to give a definition of what we mean by the term. Now the chapter which you have just read teaches you that the motion of which I have been speaking is properly measured by what has been called *momentum*, that is, by a quantity which is jointly proportional to the weight and the velocity.

P. You mean the *mass* and the velocity.

T. It is equally true that the momentum is jointly proportional to the *weight* and the velocity, because the mass varies as the weight; but my reason for using the weight in this instance is, that I wish to keep clear of all new terms which may introduce a difficulty. *Suppose* then that we regard the *motion* as being measured by a quantity jointly proportional to the weight and velocity, or by μWv, where v is the velocity, W the weight, and μ a constant quantity; then I say that the quantity of motion destroyed in the larger weight, or generated in the smaller, in a unit of time, is the measure of the tension of the string.

P. But how is μ to be found?

T. Very easily; we say that a force is to be measured

by the momentum it generates in a unit of time; for what is true of the tension of the string must be true of all forces. Now suppose the force to be the weight itself, then we know from experiment that the velocity generated in a unit of time is 32.2 or g, and the momentum generated is therefore μWg, but this is according our principle to be the measure of the weight;

$$\therefore \mu Wg = W,$$

$$\text{or } \mu = \frac{1}{g};$$

hence the expression for the momentum is $\frac{Wv}{g}$.

Now in our actual problem let g' be the velocity generated in one second, then since the whole weight in motion is 3lbs., the *momentum* generated in one second will be $\frac{3}{g}g'$; and this, therefore, must be the measure of the force which causes the motion; but this is the difference between the two weights, or 1 lb.; hence we have

$$\frac{3}{g}g' = 1,$$

$$\text{or } g' = \frac{g}{3};$$

it appears then that whereas the 2lb. weight if free would have had a velocity g generated in it in one second, as it is there is only a velocity $\frac{g}{3}$ generated, or $\frac{2g}{3}$ is destroyed, and the momentum destroyed is therefore $\frac{4}{3}$, (since in the expression $\frac{Wv}{g}$, $W = 2$ and $v = \frac{2g}{3}$); in other words, the tension of the string is $\frac{4}{3}$ lbs.

P. This mode of measuring *motion* seems to be arbitrary.

T. Undoubtedly; I do not give you this as a sufficient solution of the problem, though a probable one; I only wish you to see that we must have some relation established between the statical measure of a force which is nothing more than the pressure which will counteract it, and the dynamical measure of a force which depends upon the motion produced. There can be no doubt of the motion produced being the proper measure of a force, since it is the effect of the force, and the effect must be the measure of the cause; but the question meets us, *how is the motion to be measured?* The measure which I have assumed is (in the case of a uniform force) the momentum generated by the force in one second, understanding by momentum the quantity $\frac{Wv}{g}$. This mode of measuring a force is, as you have remarked, to a certain extent, arbitrary; but still the manner in which I have treated the problem will throw light upon the subject, and at all events will point out to you the absolute necessity of obtaining some mode of measuring force in dynamics which shall be comparable with the weight of a body. In the present instance the tension of the string manifests its action by destroying motion, and I require to know how many pounds weight are equivalent to the tension; we must, therefore, have some connecting link between motion destroyed and pounds weight.

P. And the proof of the correctness of the method which you have adopted of measuring a pressure by the momentum generated in a unit of time is, I suppose, experiment?

T. Mathematicians do not always agree amongst themselves as to what laws must be referred to experiment, and what laws may be demonstrated independently; and when laws have been established and proved in thousands of different cases to be conformable with the truth, it is a question rather curious than useful to determine the simplest basis upon which they can be made ultimately to rest. *I should* say, however, in the present case, that be-

yond doubt experiment is necessary to enable us to assert that the quantity of matter which a body contains is measured by its weight; for if we assume weight as the measure of the quantity of matter in a body, we must then ascertain that the action of gravity upon a body depends upon the *quantity*, and not the *quality* of the matter of which it is composed; and if we define bodies to have the same mass, or the same quantity of matter, when the same uniform force generates in them equal velocities in a unit of time, we must then ascertain by experiment whether this definition will coincide in its result with the statical definition of mass, that is, whether two bodies which are of equal mass dynamically, will be of equal mass statically. To ascertain this, I think we must have recourse to experiment, because we cannot assert anything *à priori* respecting the nature of the attraction of the earth, or of one particle of matter upon another.

One of the best experimental proofs of the truth of the principles developed in this chapter is that given by *Atwood's Machine.*

This machine consists, in fact, of an apparatus by means of which the problem of Art. 15. p. 89. can be exhibited experimentally. Two weights are connected by a fine silk thread which passes over a wheel the axle of which is made to turn upon friction-wheels*; by this method of suspension, and by the help of great delicacy of workmanship, it is found that the effects of friction may be almost entirely avoided. By making the two weights to differ slightly from each other, we can make the motion as slow as we please, and therefore convenient for experiment. A graduated scale is placed close to one of the weights, so that the space through which it moves can be noted; and the time of motion is observed by means of a clock made to beat seconds distinctly, so that the time can be ascertained by the ear without the necessity of looking at the clock-face. With this machine, then, the space described, and the velocity generated can be ascertained by direct observation.

* *Elementary Statics*, p. 187.

Now let W and W' be two weights of which W is the greater; then the weight which is effective in producing motion is $W - W'$, and the weight moved is $W + W'$; you will perceive that $W + W'$ can be made to remain the same while $W - W'$ is varied, that is, we can make experiment upon the effect of changing the moving force while the mass moved remains the same; and the result of the experiment is, that the accelerating force as determined by direct experiment is in all cases proportional to $\frac{W - W'}{W + W'}$; in other words,

the moving force $\propto$ mass moved $\times$ accelerating force;

and this is the result which forms the subject of the chapter which you have just now read.

P. I do not see in what manner the accelerating force is determined by experiment.

T. What we have to ascertain is the velocity generated in a given time; if the two weights be equal, there is no moving force; and if the weights be set in motion, one will rise and the other fall uniformly; now suppose that we take two weights precisely equal, and that we place upon one of them a small loose weight, then motion will ensue; after a given time let the small weight be gently removed, which may be easily arranged, then the equal weights will move uniformly; and if the space moved through in a given time be noted, this will give us the velocity generated, that is, the accelerating force.

And you will perceive that this machine also gives us a convenient method of experimenting upon the force of gravity; for instance, the fact of gravity being an uniform force may be proved by means of it; for the velocity generated in a given time may be ascertained in the manner just explained, and it can be easily shewn that under all circumstances the velocity generated is proportional to the time, that is, the motion is uniformly acce-

lerated. The difficulty in making experiments upon falling bodies consists in the rapidity with which they fall; they may be made to fall as slowly as we please by causing them to move upon an inclined plane, the method adopted by Galileo; but the method supplied by Atwood's machine is far more convenient, and susceptible of greater accuracy.

P. Is it by Atwood's machine that the accelerating force of gravity is actually found?

T. It might be so determined; for let f be the accelerating force corresponding to the two weights W and W', then (as has been stated) for different experiments,

$$f \propto \frac{W - W'}{W + W'},$$

$$\text{or } f = C \cdot \frac{W - W'}{W + W'},$$

where C is a constant quantity. Now suppose that $W' = 0$, then f becomes equal C, that is, C is the accelerating force when the weight W falls freely, or what we have before called g; hence

$$f = g \cdot \frac{W - W'}{W + W'},$$

$$\text{or } g = \frac{W + W'}{W - W'} f;$$

now in this formula f is known by direct experiment, and the proportion of W to W' is known, and hence the value of g is determined.

For example, let $W = 16$ oz., and $W' = 15$ oz., then

$$g = 31 f;$$

f must be found by removing an ounce weight from W after a given time in the manner before described, and then noting the space passed over in a given time. Thus g will be known, but its value can be determined more

exactly by means of observations of the pendulum, in a manner which you will understand hereafter.

P. And I suppose that all the results obtained concerning falling bodies might be tested experimentally by this machine.

T. Certainly; for instance, that the velocity acquired by a body in one second is measured by twice the space passed over in one second from rest, that the height to which a body will rise is proportional to the square of the velocity of projection. These and the like propositions may without much difficulty be submitted to experiment; but I think it is not necessary to trouble you with such applications of the machine; I should say that its two great uses are, to demonstrate that bodies fall with a uniformly accelerated velocity, or that gravity is a uniform force, and to demonstrate that velocity generated is in all cases proportional to the ratio of the weight producing motion to the weight moved.

P. There is one other point upon which I desire some explanation. We have been engaged in considering the mode of estimating pressures dynamically; but I confess that I should be quite at a loss if I were asked what was the dynamical measure of a pound weight.

T. The dynamical measure of a uniform pressure is, as we have seen, the momentum which it will generate in a second, and this method of estimating a force would give you the dynamical measure of a pound weight; but the weight which we call a pound is an arbitrary quantity, having no natural connexion with the quantities involved in Dynamics; nevertheless we shall find that the conventions which we have already made will make it necessary for us to take a certain dynamical unit of weight, and when we have once ascertained what that unit is, we shall then estimate all other weights accordingly.

P. What do you mean by a dynamical unit of weight?

T. In Statics you will remember that we agreed to represent a force by a letter P, understanding by P a force equivalent to P lbs.; we might have said P oz.; but pounds were taken instead, as being on the whole the most usual standard of weight; that is, we assumed 1 lb. for our statical unit. It was, in fact, perfectly indifferent what unit we assumed since we were only concerned with the ratio of one force to another, and the forces P and Q have the same ratio whether we regard them as P lbs. and Q lbs., or as P oz. and Q oz.; but in Dynamics it is not indifferent what we call unity; the unit will, in fact, be determined by the equation

$$W = Mg;$$

or rather, if we denote volume by V, and density by D,

$$W = DVg.$$

Now in this equation let us take $D = 1$, that is let us take the density to be that density by which it is usual to measure others, namely, the *density of distilled water*; let us also take $V = 1$, that is let V be a cubic foot, since a foot is always taken as the unit of linear measure, and therefore the cubic foot must be taken as that of volume or of cubic measure; with these assumptions W becomes *the weight of a cubic foot of distilled water.* A cubic foot of distilled water may be *weighed*, and it will be found to weigh 1000 oz. nearly; take this as its weight, and our equation then becomes

$$1000 \text{ oz} = g,$$

$$\text{or } \frac{1000}{32.2} = 1;$$

in other words the dynamical unit of weight is $\frac{1000}{32.2}$ oz.; hence 1 oz. dynamically is measured by $\frac{32.2}{1000}$ or .0322, and 1 lb. will be measured by 16 times this quantity.

P. Would it not have been better to have taken this dynamical unit of weight as the statical unit also?

T. There would have been no practical advantage in such arrangement; and there would have been this disadvantage, that the unit of weight being made to depend upon the accelerating force of gravity, would have varied from place to place upon the earth's surface; and this for ordinary purposes would not have been convenient. The reasons why the accelerating force of gravity varies from one point to another upon the earth's surface, we may perhaps discuss at a future time.

EXAMINATION UPON CHAPTER IV.

1. How is the *mass* of a body measured? Distinguish between the statical and dynamical measures of a *mass*.

2. What is meant by the *density* of a body? And how is density measured?

3. Define *momentum*.

4. Explain the meaning of the equation $W = Mg$.

5. Enunciate the Third Law of Motion.

6. When one weight draws up another with which it is connected by means of a fine string passing over a very small pully, determine the motion, and find the tension of the string.

7. Why is the pully in the preceding question spoken of as *very small?*

8. A weight of 4lbs. resting upon a plane inclined at an angle of 30° to the horizon is drawn up by means of an equal weight connected with it by a fine string which passes over a small pully at the summit of the plane, the second weight hanging freely; find the tension of the string.

9. A weight M lying upon a smooth table is made to move by means of a string which passing over a pully is connected with another weight M' hanging freely; find the inclination of the plane upon which M would move so that its velocity should be accelerated precisely as it actually is in the above case.

10. A weight M is drawn by another M' up an inclined plane of length a and inclination θ, as in question 8; find at what point M must have arrived in order that the string being cut it may just reach the summit of the plane.

11. How does Atwood's machine enable us to determine the relation between moving and accelerating force?

12. Shew how Atwood's machine may be used to determine the accelerating force of gravity.

13. Why is it that a ball of lead and another of cork, precisely equal in volume, will not fall to the earth with like velocity?

14. What is the dynamical unit of weight, and why is that unit taken?

15. One weight draws up another by means of a fine string passing over a pully, and it is ascertained that the string undergoes a tension equivalent to m lbs. weight; the larger weight being doubled, the tension becomes n lbs; find the two weights.

16. When two weights move as in the preceding question, shew that the centre of gravity of the two weights remains at rest during the motion.

17. Two scales are suspended by a string over a small pully; six equal bullets are placed in one scale and six in the other; shew that the tension of the string is greater with this arrangement of the bullets, than with any other.

18. If two unequal weights connected by a string be allowed to fall, the string being vertical, what will be the tension of the string?

CHAPTER V.

ON IMPULSIVE FORCE. IMPACT OF BALLS.

1. We have seen that the measure of a uniform force is the momentum generated by it in a certain given time, as one second; or if the force be estimated only as an accelerating force, then the velocity generated in one second is the measure of it. This mode of estimating force supposes therefore that we have the means of determining the effect which will be produced by a force acting during a given finite time: but suppose the case of a force of very great intensity acting during an exceedingly short period of time, as the thousandth part of a second, and suppose that the intensity of the force is so great that this very short time suffices to enable the force to generate a finite velocity, then it is manifiest that the mode of estimating the force above referred to will not serve us, because we shall have no means of ascertaining what the effect would be if this very intense force were to act during a finite time.

Take an example. A cricket-ball is delivered full pitch, it is moving in a certain direction with a finite velocity, when being struck by the bat, not only is its velocity destroyed, but a velocity is generated in the opposite direction apparently instantaneously. If we desire to find a measure of the blow which the ball receives, it is clear that we must adopt some method different from that already introduced, since we know nothing of the action of the bat beyond the phenomenon above described, and we have no means whatever of determining what the result would be if the action of the bat were conceived to take place during one second.

2. Here then we come upon a new class of forces, differing from those which we have already considered, not

in kind but in intensity, and in the length of time during which they act; this difference being sufficient to make it necessary to adopt a new mode of estimating their value. This new class of forces we term *impulsive*, denoting by the name *finite* that kind of force which is measured by the momentum generated in a unit of time or in one second. The distinction between the two will be seen from the following formal definitions.

DEF. *A finite force is one which requires a finite time to generate a finite momentum, or to generate a finite velocity in a body of finite mass.*

DEF. *An impulsive force is one which requires only an indefinitely short time to generate a finite momentum, or to generate a finite velocity in a body of finite mass.*

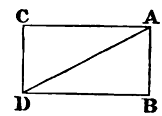
3. It will be perceived that an impulsive force defined as above is strictly speaking an imaginary thing; that is to say, every force with which we are concerned in practice does act during a finite though it may be an extremely short period of time. In fact, the action which takes place in every case of impact is of a very complicated kind, and must obviously require time for its accomplishment. Take for example the impact of two billiard-balls; they strike each other and appear to separate and fly off in opposite directions instantaneously; but the effect is in reality by no means instantaneous. When the balls first come in contact the ivory of which they are formed gives way in consequence of the blow, the balls flatten until the velocity is destroyed, then in consequence of the property of ivory which we call *elasticity*, the balls resume their form and in so doing generate a velocity in the opposite direction to that of the impact, and the balls fly apart. This is what takes place in the case of impact of elastic bodies, and the greater number of bodies are more or less elastic; and it is clear that this complicated action must occupy a finite, though of course a very short, period of time; nevertheless forasmuch as we are not able to take account of

the time so occupied, it is convenient to define an impulsive force mathematically as that which generates a finite momentum in an *indefinitely* short time; and we may consider this as a mathematical limit to which the impulsive forces with which we are concerned approach sufficiently near in character to be regarded as coming under the definition.

4. The question arises then, how are we to estimate an impulsive force? We know nothing of the law of its action; all that we know is the momentum which it actually generates; this therefore must be our measure of the impulse; that is, an impulsive force must be measured, not by the momentum which would be generated in one second, but by the whole momentum actually generated. Speaking of the accelerating effect of an impulse we should say, that the accelerating force of an impulse is measured by the velocity actually generated. Suppose, for example, that a body of mass M is observed to strike a fixed obstacle, a wall for instance, with a velocity V and to rebound with a velocity V', then the momentum $M(V-V')$ has been destroyed by the impact; and this is therefore the measure of the impulsive force with which the body strikes the wall.

5. It may be thought that a confusion will be introduced by our thus adopting two methods of measuring two kinds of force, which are in fact essentially the same. But it will be sufficient to remove all fear of such confusion to remark, that the two kinds of force, namely, *finite* and *impulsive*, can never occur in the same problem. For suppose we have a body under the action of both kinds of force, then by hypothesis the finite force cannot generate or destroy any finite amount of momentum during the time which suffices for the entire action of the impulsive forces; and therefore, so far as these latter forces are concerned, we may regard the finite forces as not acting at all. Suppose, for example, that a ball is falling vertically under the

action of gravity, and suppose that it receives a blow in a horizontal direction; at the instant of the blow let the ball's velocity be V, and let the intensity of the blow be such that if the ball were at rest the blow would cause it to move off with a velocity V'; then the actual velocity of the ball after the impact will be found by combining the vertical velocity V with the horizontal velocity V', exactly as though gravity were not acting in the body; in fact, take AB, AC, proportional to V, V', complete the rectangle $ACDB$, and join AD, then if A be the place of the body at the time of impact, it will after impact describe a parabola touching AD in A, and starting with the velocity $\sqrt{V^2 + V'^2}$. Why is not gravity taken into account? because during the very brief period of the impact it can generate no appreciable velocity. To make this more apparent let us assign an actual period of duration to the blow: suppose it to last during the hundredth part of a second, then in that time gravity generates a velocity of .322 of a foot; suppose the value of V' to be 100, then we have committed an error to the extent of omitting a quantity such as $(.322)^2$ as compared with 100^2. It will be seen that an error such as that now described is very small, and the smaller the duration of the impulse the smaller will be the error committed in neglecting the effect of gravity during the time of impact; on the mathematical supposition of an impulsive force producing its effect in an *indefinitely* short time the error altogether vanishes.



Problems of impact therefore involve impulsive forces, and no others, and the measure of each impulsive force is the whole momentum which it generates or destroys.

6. It has been before mentioned (p. 88.) that when a body is animated with a certain amount of momentum, it cannot lose any portion of that momentum without imparting to some other body, or bodies, exactly as much momentum as it loses, and that either by impact or otherwise. Suppose, for instance, that a ball of mass M has a velocity

V, and suppose that it impinges upon another ball of mass M' which is at rest, and suppose that it is observed to rebound with half its original velocity, then the amount of momentum lost is $\frac{MV}{2}$, and this must be gained by the ball M'; consequently M' will move off with the velocity $\frac{MV}{2M'}$. This law is perfectly general, and applies to all bodies whether elastic or not.

7. It may be urged as an exception to our assertion, that momentum cannot be lost by a body without the generation of an equal momentum in some other, that momentum can be destroyed by a fixed obstacle, as by a wall in the case supposed in Art. 4. The exception, however, is only apparent; for in nature there is no such thing as a fixed obstacle; even if the wall were perfectly rigid and perfectly firmly fixed to the earth, which cannot be, still the earth is not fixed, and the impact of a body upon the earth's mass generates a momentum, but no sensible velocity, on account of the immense mass upon which the impact takes place.

8. We now proceed to the solution of some problems of impact. There is only one class of problems with which we can conveniently deal, namely, problems concerning the impact of balls; the balls we shall regard as being spherical and homogeneous; and there will be two cases to consider, according as the balls are inelastic or elastic.

9. All bodies in nature are, in fact, more or less elastic; that is, if they be compressed there is a tendency, greater or smaller, to recover their original form. But the degree of elasticity varies very much; thus ivory and glass are substances of extreme elasticity, boxwood and ebony of much greater elasticity than the softer kinds of wood, as willow and deal; lead is very imperfectly elastic, and a ball made of soft clay would be almost totally devoid of elasticity.

The extreme case of actual inelasticity, although one with which we are very little concerned in practice, is so much more simple than that of elasticity in a mathematical point of view, that we shall consider it first.

10. Def. *A body is said to be inelastic when after suffering compression by impact, there is no tendency to recover its form, and consequently no impulsive force generated by the restitution of its form.*

11. Def. *The impact of two balls is said to be direct, when the motion of their centres previously to impact is in the same straight line. When this condition is not satisfied, the impact is said to be oblique.*

12. Prop. *Two inelastic balls moving in the same direction with given velocities, impinge directly upon each other; to find the velocity of each after impact.*

This problem may be solved by means of the general principle, already laid down, of the constancy of the quantity of momentum before and after impact.

For let M, M' be the masses of the two balls,

V, V' the velocities before impact.

Then after impact there will be no force tending to separate them, they being by hypothesis inelastic, and their velocity will therefore be the same; call it v.

The whole momentum before impact is $MV + M'V'$, and the whole momentum after impact is $Mv + M'v$; but these must be equal by our general principle;

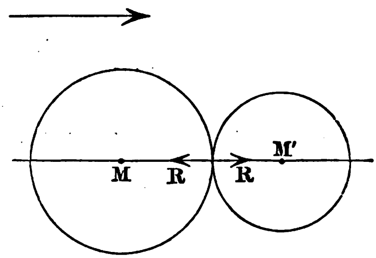
$$\therefore\ Mv + M'v = MV + M'V',$$

$$\text{or } v = \frac{MV + M'V'}{M + M'};$$

which formula gives us the common velocity of the two balls after impact.

13. It will be convenient to solve the problem in a rather different manner.

Let M, M' be the masses of the two balls, as before, and let the direction of motion be that indicated in the figure by the arrow. Then the effect of the impact is to produce an impulsive action between the two balls. This impulsive action will be equal and in opposite directions upon the two balls; this is, in fact, to say that as much momentum will be generated in one ball as is destroyed in the other.



Let V, V' be velocities before impact, as before. Then the momentum of M after impact will be $MV - R$, and that of M' will be $M'V' + R$; and the corresponding velocities will be $V - \frac{R}{M}$, and $V' + \frac{R}{M'}$. But these two velocities must be the same by the definition of inelasticity, hence,

$$V - \frac{R}{M} = V' + \frac{R}{M'},$$

$$\text{and } R = \frac{V - V'}{\frac{1}{M} + \frac{1}{M'}} = \frac{MM'}{M + M'}(V - V').$$

This expression shews the amount of momentum which passes from one ball to the other by the impact; and the common velocity after impact

$$= V - \frac{R}{M} = V - \frac{M'}{M + M'}(V - V') = \frac{MV + M'V'}{M + M'},$$

as before.

14. If we solve the problem by the former method, we

can easily find the amount of momentum which passes from one ball to the other. For the momentum lost by it is

$$MV - Mv, \text{ or } MV - M\frac{MV + M'V'}{M + M'},$$

$$\text{which} = \frac{MM'}{M + M'}(V - V'),$$

the value above obtained for R.

15. It will be observed then, that the two preceding methods differ only in this, that in one we have first found the common velocity and then the amount of momentum lost and gained, in the other we have first found the momentum and then the common velocity. It will be remembered that we solved the problem of the two weights connected by a string, in Chap. IV., by two analogous methods; and it is worth while to observe the connexion between that problem and the present one. The formulæ will be found to be much alike, and the problems are in fact very cognate in their principle, the tension of the string in the former problem being the means of communicating momentum from one body to the other, and causing them to move with the same velocity; results which in the problem just now solved are produced by the impact of the balls and the absence of all elasticity.

16. Suppose we have three balls, the masses of which are M, M', M''; let their centres be in the same straight line, and let V, V', V'' be their velocities before impact. Let M overtake M' before any impact takes place between M' and M''; then the common velocity of M and M' will be

$$\frac{MV + M'V'}{M + M'};$$

and if an impact now take place between M' and M'' the common velocity after impact will be

$$\frac{MV + M'V' + M''V''}{M + M' + M''}.$$

17. And it is easy to see that the same thing will hold of any number of balls, that is, if there be n balls $M_1 M_2 \dots M_n$ in a row, moving with velocities $V_1 V_2 \dots V_n$, and they come successively in contact, the common velocity of the balls after impact will be

$$\frac{M_1 V_1 + M_2 V_2 + \dots + M_n V_n}{M_1 + M_2 + \dots + M_n}.$$

18. It may be remarked that any ball may be accounted as proceeding in the opposite direction to that above supposed by making its velocity negative. Thus if two balls are moving in opposite directions with velocities V, V', the common velocity after impact will be

$$\frac{MV - M'V'}{M + M'},$$

and the motion will be in the direction of that of M or that of M', according as MV is greater or less than $M'V'$.

19. It may also be remarked that if V and V' be the velocities of two bodies M and M', which are moving uniformly in the same straight line, then the velocity of the centre of gravity of the two is $\dfrac{MV + M'V'}{M + M'}$. This follows at once from the general formulæ for the position of the centre of gravity investigated in the Treatise on Statics, (Chap. III. Art. 14); for let x be the distance of the centre of gravity of M and M' from the point which it occupied at any given time, a, a' the distances of the centres of M and M' from the same point, then

$$x = \frac{Ma + M'a'}{M + M'}:$$

now suppose a and a' each to increase uniformly, one at the rate of V feet per second, the other at the rate of V' feet per second, then x increases at the rate of

$$\frac{MV + M'V'}{M + M'}$$

feet per second; in other words, the centre of gravity moves with a velocity

$$\frac{MV + M'V'}{M + M'}.$$

Hence it appears, that *the velocity of the centre of gravity of the balls is the same after impact as before;* for this quantity which expresses the velocity of the centre of gravity before impact has been shewn to express the velocity of either of the balls, and therefore of their centre of gravity after impact.

And the same thing holds true of any number of balls.

Hence, *if the centre of gravity be at rest at any moment it will always continue at rest.*

20. This remarkable proposition has been proved by reference to the formulæ investigated for inelastic balls; it is however true whether the balls be inelastic or no; in fact, it is a perfectly general proposition flowing at once from the general principle of the constancy of the momentum of the bodies.

For, let there be any number of bodies $M, M', M'', \ldots$

and let their velocities before impact be $V, V', V'', \ldots$

.... after .... $v, v', v'' \ldots$

Then the whole momentum before impact is

$$MV + M'V' + M''V'' + \ldots;$$

and the whole momentum after impact is

$$Mv + M'v' + M''v'' + \ldots;$$

and by our general principle these two momenta must be the same;

$$\therefore MV + M'V' + M''V'' + \ldots = Mv + M'v' + M''v'' + \ldots;$$

and this equation expresses the fact that the velocity of the centre of gravity of the balls before impact is the same as the velocity after impact.

21. We now proceed to explain the method of treating elastic bodies. When two elastic balls impinge directly upon each other, a compression of the form of the balls takes place, and so long as this compression continues the nature of the action between them is the same as if they were inelastic; the difference is, that whereas inelastic bodies after compression remain compressed, elastic bodies recover their form; it will be convenient then to consider the impact as consisting of two parts, which we will call *compression* and *restitution*. Now when a ball M impinges upon another M', a certain portion of M's momentum is destroyed during the compression, and another portion during the restitution; the quantity destroyed during the compression is precisely that which would be destroyed if the bodies were inelastic, since until restitution commences the peculiar feature of elasticity does not come into play; hence the quantity destroyed during compression is known, and will, according to our former notation, be equal to $\dfrac{MM'}{M+M'}(V - V')$. In order to determine the momentum destroyed during restitution we must refer to experiment, and by that means we deduce this very simple rule, that *the momentum destroyed in restitution bears to that destroyed in compression a constant ratio*, that is, a ratio independent of the intensity of the impact or any such circumstances, and *depending only upon the nature of the substance of which the balls are made.* Let us call this ratio e, then the momentum destroyed in restitution will be $e\dfrac{MM'}{M+M'}(V - V')$; and therefore the whole momentum destroyed by the impact will be

$$(1+e)\frac{MM'}{M+M'}(V - V').$$

Hence, if v, v' be the velocities of M and M' after impact, we have

$$Mv = MV - (1 + e)\frac{MM'}{M + M'}(V - V');$$

and since the momentum lost by M is gained by M', we have also

$$M'v' = M'V' + (1 + e)\frac{MM'}{M + M'}(V - V').$$

22. The quantity e is called the *modulus of elasticity*, and is (as we have said) dependent only upon the nature of the substance of which the balls are made: thus it has a certain value for ivory, another for glass, another for box-wood, and so on. It may be observed that elasticity cannot be predicated of one substance only, but depends upon the nature of both substances in case two impinging bodies are not composed of the same; however highly elastic one of the substances may be, its elasticity may be reduced to any extent by the want of elasticity on the part of the other impinging body; thus ivory will not indicate any considerable elasticity if it impinge upon clay.

The quantity e is always in practice less than 1; we may conceive of the case in which it actually becomes equal to 1, in other words in which as much momentum is generated or destroyed in restitution as in compression, but no elastic substance ever attains to this limit; problems are however sometimes made for convenience sake upon this imaginary hypothesis, and bodies for which the supposition is made are said to be *perfectly* elastic.

23. If according to the method adopted in Art. 13. we denote the impulsive force of compression by R, and if we in like manner denote the impulsive force of restitution by R', then we may say that the result of experiment is to shew that

$$R' = eR;$$

so that the whole impulsive action is $R + R'$ or $(1 + e)\ R$. Hence we can put the rule for treating the impact of elastic bodies in a very convenient form by saying; *first solve the problem upon the hypothesis of inelasticity, and find the impulsive action* R, *and then the impulsive action required will be*

$$(1 + e)\ R.$$

24. We will illustrate this rule by applying it to the case (already solved) of two balls M, M' impinging directly.

First suppose them inelastic, and let R be the impulsive action between them; then

$$\text{velocity of } M \text{ after impact} = V - \frac{R}{M},$$

$$\text{............ } M' \text{ ....} = V' + \frac{R}{M'};$$

but these two must be equal upon our present hypothesis,

$$\therefore\ V - \frac{R}{M} = V' + \frac{R}{M'},$$

$$\text{or } R = \frac{MM'}{M + M'}(V - V').$$

Now let v, v' be the velocities of M, M' after impact; then by our rule,

$$Mv = MV - (1 + e)\ R,$$
$$M'v' = M'V' + (1 + e)\ R,$$

where R has the above value.

25. We have said that the physical fact, expressed by the formula $R' = eR$, is the result of experiment; it will be easily understood that the momentum generated or destroyed cannot be the immediate subject of experiment; that which we can observe directly is the rate at which the balls approach each other before impact and recede from each other after impact; in other words, we can observe the *relative* velocities before and after impact. Let as before V, V' be the velocities before impact, v, v' the velocities

after impact; then, e being such as we have before described, it is observed that

$$v' - v = e(V - V'),$$

in other words, that *the relative velocity after impact bears a constant ratio to the relative velocity before impact.* Let us see how from this experimental result we can shew that $R' = eR$.

Now we have from our general principles,

$$Mv = MV - (R + R'),$$

$$M'v' = M'V' + (R + R');$$

$$\therefore\ v' - v = V' - V + (R + R')\left(\frac{1}{M} + \frac{1}{M'}\right),$$

$$\text{or } (1 + e)(V - V') = (R + R')\frac{M + M'}{MM'};$$

$$\therefore\ R + R' = (1 + e)\frac{MM'}{M + M'}(V - V');$$

but from the investigation for the case of inelastic balls we know that

$$R = \frac{MM'}{M + M'}(V - V'),$$

$$\therefore\ R' = e\frac{MM'}{M + M'}(V - V') = eR.$$

Hence it follows from the observed relation between the relative velocities before and after impact, that the momentum generated or destroyed during compression bears a constant ratio to that generated or destroyed during restitution.

26. Prop. *An elastic ball impinges directly upon a perfectly hard fixed plane; to find the velocity after impact.*

Let M be the mass of the ball; V the velocity before impact; v the velocity after impact; R the impulsive

action between the ball and the plane as the hypothesis of inelasticity.

Then, upon this hypothesis, there is no motion after impact,

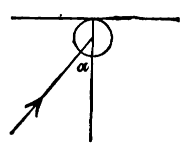
$$\therefore\ 0 = MV - R;$$

$$\text{hence } Mv = MV - (1 + e)\,R,$$
$$= -eR = -eMV,$$
$$\text{or } v = -e\,V;$$

that is, the velocity after impact is less than before in the proportion of e to 1; the minus sign indicates that the motion after impact is in the direction opposite to that before impact, which must manifestly be the case.

27. The problem of oblique impact requires no principles in addition to those already explained. When impact takes place obliquely, we have only to resolve the impulsive action into two parts, one normal to the balls, the other tangential; the former will be treated according to the same laws as direct impact, the latter will be unaffected since we suppose the balls to be smooth. The method will be understood from the following applications.

28. Prop. *An elastic ball impinges upon a perfectly hard plane, the direction of the ball's motion before impact making a given angle* (a) *with the normal to the plane; required the motion after impact.*



Let M be the mass of the ball;

V its velocity before impact;

v .... after ....

θ the angle which the direction of motion after impact makes with the normal to the plane;

then before impact the velocity parallel to the plane is $V \sin a$, and that perpendicular to the plane is $V \cos a$; and the corresponding quantities after impact are $v \sin \theta$, and $v \cos \theta$.

Hence, by Art. 26, we shall have

$$v \cos \theta = -eV \cos \alpha,$$

and since the velocity parallel to the plane is not affected,

$$v \sin \theta = V \sin \alpha.$$

Adding the squares of these two equations,

$$v^2 = V^2 (\sin^2 \alpha + e^2 \cos^2 \alpha)$$
$$= V^2 - (1 - e^2) V^2 \cos^2 \alpha.$$

Hence the velocity will be diminished, as might have been expected, except in the extreme case of perfect elasticity, for which $e = 1$.

For the direction of the motion, we have

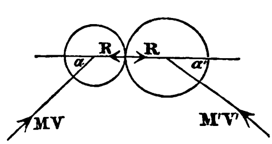
$$\cot \theta = -e \cot \alpha.$$

The minus sign shews that the ball will rebound on the *opposite* side of the normal, and since e is less than unity, θ will be greater than α, except in the extreme case of perfect elasticity when the two are equal.

29. Let us now take the more general case of two balls impinging obliquely upon each other. We will suppose them to be moving in opposite directions, contrary to the supposition made hitherto.

PROP. *Two elastic balls impinge obliquely upon each other; to determine the motion after impact.*

Let V, V' be the velocities before impact, α, α' the angles which the directions of motion make with the straight line joining the centres of the balls at the instant of impact; v, v', θ, θ', the corresponding quantities after impact.



Let us first find the impulsive action R upon the hypothesis of the balls being inelastic, (Art. 23, p. 118.) Then

treating the problem, so far as the direction of the common normal or line joining the centres is concerned, as if the impact were direct, we have

$$V\cos\alpha - \frac{R}{M} = \frac{R}{M'} - V'\cos\alpha';$$

$$\therefore\ R = \frac{MM'}{M+M'}(V\cos\alpha + V'\cos\alpha').$$

Hence by our principles,

$$Mv\cos\theta = MV\cos\alpha - (1+e)R$$

$$= MV\cos\alpha - (1+e)\frac{MM'}{M+M'}(V\cos\alpha + V'\cos\alpha') \ldots (1);$$

and since the motion in the direction perpendicular to the common normal is not affected by the impact, we have

$$Mv\sin\theta = MV\sin\alpha \ldots\ldots\ldots\ldots (2).$$

In like manner for the other ball,

$$M'v'\cos\theta' = (1+e)\frac{MM'}{M+M'}(V\cos\alpha + V'\cos\alpha')$$

$$-M'V'\cos\alpha' \ldots\ldots (3).$$

$$M'v'\sin\theta' = M'V'\sin\alpha' \ldots\ldots\ldots (4).$$

These four equations entirely determine the motion of the two balls.

30. We may vary the problem in many ways. Suppose, for instance, that the ball M' is at rest, which will give us the case of a billiard-ball struck by another. Then making $V' = 0$, we have

$$Mv\cos\theta = MV\cos\alpha - (1+e)\frac{MM'}{M+M'}V\cos\alpha,$$

$$Mv\sin\theta = MV\sin\alpha,$$

$$M'v'\cos\theta' = (1+e)\frac{MM'}{M+M'}V\cos\alpha,$$

$$M'v'\sin\theta' = 0;$$

from this last equation we have $\theta' = 0$, as might have been foreseen; to simplify the formulæ still further, let us suppose the balls equal, or $M = M'$, and let $e = \frac{4}{5}$, which is nearly the value which it has for ivory; then

$$v \cos\theta = V\cos a - \tfrac{9}{10} V\cos a = \tfrac{1}{10} V\cos a,$$

$$v\sin\theta = V\sin a,$$

$$v' = \tfrac{9}{10} V\cos a;$$

$$\therefore\ v^2 = V^2\left(\sin^2 a + \frac{\cos^2 a}{100}\right) = V^2(1 - .99\cos^2 a),$$

and $\tan\theta = 10\tan a$;

the former of these two formulæ gives the velocity of the impinging ball, and the latter shews that the direction of the motion will be changed so as to make with the direction of the common normal an angle much larger than before, but less than a right angle.

31. A very curious result will follow from the supposition that the two balls are made in such a manner that the ratio of their weights, or masses, shall be the modulus of elasticity, if, for instance, we have two ivory balls, one of which weighs four fifths as much as the other.

In general let us introduce into the equations of Art. 29, the condition that $e = \frac{M}{M'}$, or that $1 + e = \frac{M + M'}{M'}$;

then the equations (1) (2) (3) (4) become,

$$Mv\cos\theta = MV\cos a - M(V\cos a + V'\cos a')$$
$$= -MV'\cos a',$$

$$Mv\sin\theta = MV\sin a,$$

$$M'v'\cos\theta' = M(V\cos a + V'\cos a') - M'V'\cos a',$$

$$M'v'\sin\theta' = M'V'\sin a';$$

or, if the ball M' be at rest,

$$\left.\begin{aligned} v\cos\theta &= 0,\\ v\sin\theta &= V\sin a,\\ v'\cos\theta' &= eV\cos a,\\ v'\sin\theta' &= 0.\end{aligned}\right\}\ \cdots\cdots\cdots\cdots\ (A).$$

From these equations we perceive that $\theta = 90^0$, and $\theta' = 0$, and that

$$v = V \sin \alpha,$$

$$v' = e V \cos \alpha.$$

It appears then, that if the weights of the balls be adjusted as here supposed, the two balls will fly off after impact in directions at right angles to each other, whatever be the direction of impact. There is, however, an exception; if the impact be direct, it is manifest that the balls cannot move as here described; let us see how this exception is taken into account by our equations.

Resuming equations (A) it will be seen that in the case of direct impact $\alpha = 0$, hence the first two equations become

$$v \cos \theta = 0, \ v \sin \theta = 0,$$

and from these we must conclude, not that $\theta = 90^0$ as before, but that $v = 0$; hence in the case of direct impact the impinging ball is brought to rest, and the ball which was previously at rest moves off with a velocity eV.

32. We shall conclude this Chapter with two propositions, which will require the definition of a term not hitherto introduced.

Def. *The mass of a body in motion multiplied by the square of its velocity is called the* vis viva *of the body.*

Thus it may be stated, that in the case of a falling body, the *vis viva* generated in falling through a given space is proportional to the space. The student will observe that the term *vis viva* is merely used as a convenient term for expressing briefly a particular combination of mass and velocity which not unfrequently occurs in Dynamics.

33. Prop. *In the direct impact of perfectly elastic balls, the* vis viva *is the same after impact as before.*

In the case of perfect elasticity $e = 1$, hence we can easily obtain the equations, (Art. 21, p. 117).

$$\left.\begin{aligned} Mv + M'v' &= MV + M'V', \\ \text{and } v' - v &= V - V'; \end{aligned}\right\}$$

$$\left.\begin{aligned} \text{or } M(v - V) &= -M'(v' - V'), \\ \text{and } v + V &= v' + V'; \end{aligned}\right\}$$

multiplying together the corresponding sides of these equations, there results,

$$M(v^2 + V^2) = -M'(v'^2 - V'^2),$$

$$\text{or } Mv^2 + M'v'^2 = MV^2 + M'V'^2,$$

which proves the proposition.

The same proposition is true when a perfectly elastic ball impinges directly upon a perfectly hard plane.

34. Prop. *In the direct impact of imperfectly elastic balls,* vis viva *is lost by impact; that is, the* vis viva *after impact is less than it was before.*

In this case our equations are

$$Mv + M'v' = MV + M'V',$$

$$\text{and } v' - v = e(V - V');$$

$$\therefore (Mv + M'v')^2 = (MV + M'V')^2,$$

$$\begin{aligned} \text{and } MM'(v - v')^2 &= e^2 MM'(V - V')^2 \\ &= MM'(V - V')^2 - (1 - e^2)MM'(V - V')^2; \end{aligned}$$

$\therefore$ by addition,

$$\begin{aligned} (M + M')(Mv^2 + M'v')^2 = (M + M')(MV^2 + M'V')^2 \\ - (1 - e)^2 MM'(V - V')^2, \end{aligned}$$

$$\text{or } Mv^2 + M'v'^2 = MV^2 + MV'^2 - (1 - e^2)\frac{MM'}{M + M'}(V - V')^2,$$

which proves the proposition, since e is less than 1.

By making $e = 1$ we obtain the proposition of the preceding Article.

35. The preceding propositions concerning *vis viva* may be still further generalized; indeed, it is true in the most general way possible that *vis viva* is always lost by collision of bodies in whatever manner the impact may take place. It will be seen by reference to the formula of Art. 28, that the proposition is true in the case of oblique impact upon a fixed plane, for we have

$$Mv^2 = MV^2 - (1 - e^2)\, MV^2 \cos^2 a;$$

and it may also be shewn to be true in the most general case which we have considered, namely, that of the oblique impact of two balls.

For it will be seen by reference to the equations (1) (3) of Art. 29, compared with those of Art. 21, that the process adopted in the preceding Article would give us,

$$Mv^2 \cos^2 \theta + M'v'^2 \cos^2 \theta' = MV^2 \cos^2 a + M'V'^2 \cos^2 a'$$
$$- (1 - e^2) \frac{MM'}{M + M'} (V \cos a + V' \cos a')^2;$$

also from equations (2) and (4) of Art. 29, we have

$$Mv^2 \sin^2 \theta = MV^2 \sin^2 a,$$
$$\text{and } M'v'^2 \sin^2 \theta' = M'V'^2 \sin^2 a';$$

now add these three equations together, and we have

$$Mv^2 + M'v'^2 = MV^2 + M'V'^2$$
$$- (1 - e^2) \frac{MM'}{M + M'} (V \cos a + V' \cos a')^2,$$

which equation proves the proposition.

CONVERSATION UPON THE PRECEDING CHAPTER.

P. If you will allow me to begin at the end of the Chapter in asking questions, I should like to know how it came to pass that Mv^2 should have been called the *vis viva* of a body.

T. The name is a monument of an error in dynamical principles once maintained by mathematicians of great name; they held that a force which produced only a tendency to motion differed in kind from a force actually producing motion, the former they proposed to term a *vis mortua,* and the latter a *vis viva.* Thus they considered that in a body actually falling there was a force which did not exist before the body began to fall; and they were led to imagine that this *living force* was to be measured by the square of the velocity. The source of this error in measuring force was fundamentally this, that they considered the force as measured by its effect, while the body passed through a given *space,* instead of its effect during a given *time;* in the case of falling bodies you will remember that the *momentum* generated varies as the time, the *vis viva* as the space described; if then we considered the effect of a force while the body passes through a given space, we should be led to take *vis viva* as the measure of force.

P. Do you consider it manifest that this notion of a distinction between the force, in the case of a body at rest, and a body in motion, is incorrect?

T. Not manifest, because men of the first order of intellect fell into the mistake; to us it perhaps seems strange that it should not have been perceived that it was the time of a force's action, and not the space through which the body moved, upon which the effect depended; but the notion of some new property being developed in a body by motion seems not an unnatural one to strike the mind in the infancy of science.

P. I think it does not; indeed, it seems to me hard to realize that the force acting upon a body at rest and in motion can be the same.

T. There is but one measure of force, and that is the momentum generated or destroyed by it in a given

time; suppose a body to rest upon a plane, then the plane by preventing motion destroys a certain amount of momentum in each second, which would have been generated if the plane had not been there, and this amount of momentum measures the weight of the body; if the plane be removed, momentum is actually generated, and the momentum generated in one second measures the weight. Referring to symbols, I must remind you that whether a body is at rest or in motion, Mg is taken as the symbol of its weight; and in the case of a body descending upon an inclined plane, we had one resolved part of the weight producing pressure, and the other producing motion; but the weight was represented by the same symbol Mg in both cases.

P. So that in fact the true dynamical principle is, that in all cases momentum generated or destroyed is the proper measure of force.

T. Yes; that is the true basis of dynamics; and the only thing to be remarked is, that in the case of finite force we consider the momentum generated in a unit of time, in the case of impulsive force we consider the momentum actually generated during the impact. But there is no difference of principle in the mode of treating the two kinds of force, and this perhaps will appear to you clearly from Newton's method of stating the Third Law of Motion. He states that Reaction is always equal and opposite to Action; that is, that the actions of two bodies upon each other are always equal and opposed to each other in direction. "Actioni contrariam semper et æqualem esse reactionem: sive corporum duorum actiones in se mutuo semper esse æquales et in partes contrarias dirigi." And Newton gives three illustrations of the meaning of his law; first, a stone being pressed by the finger, in which case the finger is equally pressed by the stone; secondly, a horse dragging a burden by means of a rope, in which case the connecting rope causes the horse

to be drawn towards the burden just as the burden is drawn towards the horse; thirdly, a body impinging upon another, in which case any change of motion in one body will cause an equal and opposite change of motion in the other. Now here we have three kinds of action and reaction; the first statical, the second dynamical but the force finite, the third dynamical but the force impulsive; and the same principle applies to all, but the question of course is still open, how the action and reaction are to be measured, and Newton takes care to explain that the momentum generated or destroyed is the proper measure; and bearing in mind that this is the case, you will find the formula, *action and reaction are always equal and opposite,* to be a useful law to bear in mind; without knowing this great principle concerning momentum the law would be very little better than a truism, and equivalent to saying that the cause is measured by the effect.

P. It would be easy to verify the law by experiment in the case of impact; would it not?

T. Yes; and it has been done. Newton made experiments with great care and on a large scale, by means of balls suspended by strings; two balls being thus suspended in such a manner as to be in contact when at rest, one was drawn out of its position and being let go impinged upon the other; the velocity with which the ball impinged was known from the height through which it fell, and that of the ball struck by observing the height to which it rose, and it was found that in all cases the motion lost by one ball was gained by the other. In conducting these experiments it was necessary to take account of the effect of the air in resisting the motion of the balls, but I will not trouble you with an explanation of the method which Newton adopted.

You may regard it therefore if you please as an experimental fact, that in the case of the impact of bodies the momentum lost by one body is gained by the other, or

that the whole momentum before impact and after impact is the same.

And still more generally, if you have a system of bodies which act upon each other in any way either by mutual attraction, or by strings, or by impact, the whole momentum of the system never varies; and from this I may mention that it follows, that the centre of gravity of such a system if once at rest is always at rest.

P. How does that appear?

T. You see how it appears in the case considered in Art. 19; what I have said to you just now is only a generalization of the principle contained in that Article; I cannot give you the mathematical proof of the proposition in its most general form, but you will, I think, have no difficulty in perceiving that the principles which proved the proposition in that Article will establish the more general proposition which I have just now enunciated.

P. How if the centre of gravity be not at rest?

T. Then it requires very little argument to shew that it will move uniformly in a straight line, for there being no tendency to move arising from the system itself, whatever motion the centre of gravity has it will always have, precisely after the manner of a particle acted upon by no forces, and which therefore obeys the first law of motion.

Hence I may state to you, that constancy of momentum is equivalent to uniform rectilinear motion or rest of the centre of gravity.

I have endeavoured to lead you up to this general proposition, in order that you may see something of the greatness of the results which we can obtain by mathematical reasoning without any complicated calculations or mysterious symbols. The case which I have just described holds in fact in the case of the solar system; here we have the sun and the planets mutually acting upon each other, and we may at once conclude that, putting aside the con-

sideration of external influences, the action of the fixed stars for instance, the centre of gravity of the solar system is either at rest or moving uniformly in a straight line.

P. I thought that the sun was at rest, and that the planets revolved about it.

T. So it is often stated, but it is not strictly true; on account of the enormous mass of the sun the distance of the centre of gravity of the whole system from the sun's centre is not great, and therefore we commit no great geometrical error in speaking of the sun's centre being at rest when we should say the centre of gravity of the solar system; but we commit a very great dynamical error if we conceive of the sun as having any fixity of character which does not belong to the other heavenly bodies; there is no such thing as a fixed particle of matter in the universe, and dynamically speaking it would be quite as correct to say that the sun goes round the earth as that the earth goes round the sun, the fact being that neither is at rest, but that both are moving about the common centre of gravity of the whole system.

We have however been led away from the subject of impact. Have you any further question to ask?

P. I should wish to know whether there is any means of ascertaining that the action described in Art. 3, as taking place between elastic bodies actually does take place; it seems to me difficult to conceive of such an action in the case of such substances as glass or ivory.

T. Remember that however hard a substance may appear it is not rigid, and there is no reason why the form of a body should not be changed by impact and why the internal action of the particles of the body upon each other should not restore the form as soon as the pressure is removed. You may however easily satisfy yourself of the fact of the change of form thus; cover a marble table with a very delicate coating of oil, and allow an ivory ball to fall from some height above the table upon it, then it will

be found that the circular mark left upon the table by the ball will be perceptibly larger than if the ball had been merely placed gently upon the table, thus shewing incontestably that a larger portion of the surface of the ball is brought into contact by impact than otherwise, and therefore that there must be a change of form. There will be no perceptible change of form either in the ball or table after the impact, which shews that a restitution of form has taken place.

Try the experiment with a leaden ball upon a leaden table and the result will be very different; you will have a change of form by impact as before, but only an imperfect restitution of form; the ball will not rebound from the table, and the effect of the impact will be visible afterwards both upon the table and the ball.

P. That appears decisive. Will the principles which have been explained in this Chapter solve all the problems of the impact of billiard balls?

T. There is an important circumstance in the case of a billiard table which has not been taken into account, namely the effect of friction upon the balls. In consequence of this a variety of effects may be produced by a skilful player, which cannot be accounted for upon the principles of this Chapter; the actual solution of the problem of the impact of billiard balls when the friction of the table is taken into account belongs to a much more difficult portion of Dynamics, namely, that which treats of the motion of rigid bodies under the action of any forces. For though the balls, the motion of which has been discussed, are in fact rigid bodies, still in the particular case considered the forces upon any one of them all act through a single point, namely, the centre; and therefore the motion of the centre is precisely the same as that of a particle, and we have no occasion for any new principles in addition to those which apply to the motion of a particle.

P. But would the introduction of friction make any material difference?

T. A very great difference, because it would introduce the consideration of the *rolling* of the balls; and the complication introduced by this circumstance is such as to remove the problem into quite another region of Dynamics. The only way to make experiments upon the impact of elastic balls upon the principles of this Chapter is to suspend them by strings, and to let them impinge by falling against each other.

EXAMINATION ON CHAPTER V.

1. Distinguish between *finite* and *impulsive* force.

2. Distinguish between *elastic* and *inelastic* bodies; what is meant by the term *modulus of elasticity?*

3. Two inelastic balls impinge directly upon each other with given velocities, determine the velocity of each after impact.

4. Three equal inelastic balls lie with their centres in the same straight line upon a smooth horizontal table; the first being made to move with a given velocity, find the velocity which will be communicated by impact to the third.

5. Two inelastic balls impinge obliquely upon each other, with given velocities; find the motion after impact.

6. Explain generally the method of solving problems concerning the motion of impinging elastic bodies.

7. Two balls of given elasticity impinge directly upon each other; determine the motion after impact.

8. When a perfectly elastic ball impinges obliquely upon a fixed plane, the angle of reflexion is equal to the angle of incidence.

9. Two imperfectly elastic balls impinge upon each other obliquely; determine the motion after impact.

10. From the result of the preceding problem deduce the motion after impact of an imperfectly elastic ball which impinges obliquely upon a fixed plane.

11. In the impact of perfectly elastic balls no *vis viva* is lost.

12. In the impact of imperfectly elastic balls *vis viva* is lost by the impact.

13. A particle moving along a smooth horizontal plane impinges

upon a plane inclined at any angle θ to the horizon; if the particle be inelastic, find the velocity in order that the particle may rise upon the inclined plane to a vertical height h; if the elasticity be e, what will be the motion of the particle?

14. A perfectly elastic body is projected from a point in a plane inclined at an angle α to the horizon; determine the angle at which it must be projected, so that after striking the plane it may be reflected vertically upwards.

15. Determine the motion of the body in the preceding problem after it has again struck the plane.

16. Two elastic balls A and B, such that the mass of A is three times as great as that of B, are placed on a horizontal table. A impinging on B at rest drives it perpendicularly against a hard vertical plane, and it meets A in returning at half its original distance. Find the modulus of elasticity.

17. If two perfectly elastic balls, the masses of which are in the ratio of 1 : 3, meet directly with equal velocities, the larger one will remain at rest.

18. A body of given elasticity is let fall from a given height upon a plane of given inclination; find the latus rectum of the parabola described by the body after the impact.

CHAPTER VI.

ON THE CONSTRAINED MOTION OF A PARTICLE. OSCILLATION OF A CYCLOIDAL PENDULUM. APPLICATION OF THE PENDULUM.

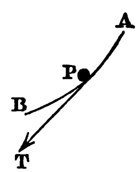
1. THERE are two ways in which we may conceive the motion of a particle to be constrained; it may either be connected with another particle, or with a fixed point by a fine thread, or it may be compelled to move in a tube, or upon the surface of a fixed curve. We have already had an instance of each of these kinds of constraint; the motion of two weights connected by a thread passing over a pully was an instance of the former, the descent of a particle upon an inclined plane of the latter. We now proceed to consider the subject more generally, premising that the mathematics supposed to be at our command are not sufficient to enable us to pursue the subject to any very great length.

2. In general, when a particle moves upon a curve, we may suppose the force acting upon it to be resolved into two parts, one in the direction of the *tangent* to the curve, the other in that perpendicular to the tangent or the direction of the *normal;* the former part will tend to accelerate or retard the body's motion, the latter will have no effect upon the velocity, but will be entirely expended in increasing or diminishing the pressure upon the curve. In the simple case of a heavy particle moving upon an inclined plane, it will be remembered, (Art. 26, p. 40) that each of these resolved parts was a constant quantity, and consequently the motion was uniformly accelerated; there would be in this case also (though this was not proved) a constant pressure upon the plane; in general such will not be the case: suppose, for instance,

that a particle runs down any curve in a vertical plane, as an arc of a circle, then the resolved part of gravity in the direction of the tangent changes from point to point, as does also that in the direction of the normal, consequently the velocity is not uniformly accelerated, neither is the pressure constant.

3. We say that the pressure is not constant; it must not be supposed, however, that the variation of the pressure arises entirely from the change of the value of the resolved part of the force in the direction of the normal, for the pressure upon the curve is only partly due to this force; this is a matter requiring much consideration, and which we will explain as clearly as we can.

Let P be a particle moving upon the inside of a circular arc AB; and let us suppose either that there is no force acting upon P, or that the whole of such force is in the direction of the tangent PT; then although there is no force in the direction of the normal to the circle, there will nevertheless be a pressure between the particle and the curve; this appears at once from the consideration, that if the curve were removed, the particle P would move in a straight line in the direction which its motion had at P, that is, it would move in the direction of the tangent PT; but the curve prevents it from doing so, and it cannot thus change the direction of the particle's motion without pressing the particle and itself sustaining a pressure. If then a particle move upon a curve in such manner that the whole extraneous force acting upon it is either in the direction of the tangent or else zero, there will still be a pressure upon the curve; and hence in the general case, in which a body is acted upon by a force partly tangential and partly normal, the pressure is due only in part to the normal force. And it is easy to see that the pressure due to the normal force may be either increased or diminished by the motion, according to circumstances; suppose, for instance, that a



heavy particle is made to run down the exterior surface of a circle in a vertical plane, then it is clear that since the body has a tendency at each moment to proceed in the direction of the tangent, the effect of the motion is to *diminish* the pressure upon the curve; on the other hand, if the body run down upon the interior surface of a vertical circle, the same considerations shew that the motion increases the pressure upon the curve. In illustration we will mention a result which will be afterwards proved, namely, that if the particle in the last supposed case run down from the extremity of the horizontal diameter, the pressure when it passes the lowest point will be three times its weight, that is, three times what it would have been if we had only taken into account the direct effect of gravity.

The principle is precisely the same if we suppose the body to be constrained by a string; for instance, if we suspend a body by a string, and make it oscillate, the body will be constrained to move in a circle exactly as though it were moving upon a circular arc, and the *tension* of the string will take the place of the *pressure* upon the curve. Thus, to take our preceding illustration, if a body be suspended by a string, and having been held so that the string shall be horizontal, be then let go, the weight which the string will have to support when it comes into the vertical position will be three times the weight of the body.

In common language the pressure or tension spoken of is said to be due to *centrifugal force;* this name is not a very suitable one, since, as we have explained, the very peculiarity of the pressure or tension in question is that it is produced by the tendency of the body to move in a certain way, which tendency is checked, and not by the immediate action of any force upon the body. The name is however generally used, and will not lead to any misconception if the student bear carefully in mind the explanations already given. We will now give a formal

statement of the meaning of the term, which will embody in a small compass what has been said.

4. PROP. *To explain what is meant by the term* centrifugal force.

When a particle moves under the action of any force, it has a tendency at each instant to continue to move in the manner in which it is moving at that instant, as regards both direction and velocity. This is an immediate consequence of the First Law of Motion. If therefore the motion of the body be curvilinear, there is a tendency at each instant to move in a direction different from that which the body is compelled to pursue; the force necessary to counteract this tendency is the measure of what is called the *centrifugal force.*

5. Let us now discuss an actual case of motion which will throw light on what precedes, and which is on other grounds important. The case is that of a particle revolving in a circular tube, or at the extremity of a thread; we will suppose no force to act upon the particle, or (which comes to the same thing) that a particle is set in motion within a circular tube which is held in a horizontal position; in the latter case it is clear that the weight of the particle will not affect its motion, and that putting friction out of the question, it will revolve uniformly for ever. Our purpose is to find the pressure upon the circular tube, or the tension of the string, according as we adopt one form of the problem or the other; the two forms are in principle identical, but the former will be more convenient for our method of treatment; we will therefore state the problem as follows.

6. PROP. *A particle of given mass revolves uniformly with a given velocity in a circular tube; to find the pressure upon the tube.*

We shall arrive at the result by first supposing the particle to move in a tube of the form of a regular polygon

having a great many sides, just as in works on Trigonometry the area of a circle is found by first investigating the area of a polygon and then supposing the number of sides of the polygon to be indefinitely great. Suppose the particle to be projected with the given velocity along one side of the polygon, then when it arrives at the end of that side or at the angular point, an oblique impact will take place upon the next side; suppose the momentum of the particle to be resolved into two parts, one in the direction of the next side of the polygon and the other perpendicular to it, then this latter portion will be destroyed by the impact, and it is the destruction of such momenta at successive impacts which constitutes the whole pressure on the curve.

Let V be the velocity, M the mass of the particle, and let the polygon have n sides; then the angle between two successive sides is $\frac{2\pi}{n}$, (Euc. I. 32, Cor. 2), and the momentum destroyed by the impact is therefore $MV \sin \frac{2\pi}{n}$. Now in a complete circuit there will be n times this amount of momentum destroyed, or $nMV \sin \frac{2\pi}{n}$; and hence the amount of momentum destroyed in one revolution is the value of the preceding expression when n is indefinitely great, or $2\pi MV$. Also since the body moves uniformly the time of a revolution will be $\frac{2\pi r}{V}$, where r is the radius of the circle; and therefore the momentum destroyed in one second

$$= 2\pi MV \div \frac{2\pi r}{V} = \frac{MV^2}{r}.$$

This then is the expression for the pressure upon the tube.

7. If we suppose the body to be attached to one end of a string the other end of which is fixed, then the tension of the string will in like manner be measured by $\frac{MV^2}{r}$.

And the tension of the string may be compared with the weight of the body; for let T be the tension, W the weight; then $W = Mg$, and

$$\frac{T}{W} = \frac{V^2}{gr};$$

or we may express the result in terms of the time required by the body to make one revolution; let P be the time, then $VP = 2\pi r$,

$$\therefore \frac{T}{W} = \frac{4\pi^2 r}{P^2 g}.$$

8. Let us illustrate the preceding formulæ by a few examples.

Ex. 1. A weight of 1 lb. revolves at the extremity of a string 1 foot long and makes a revolution in 1 second; find the tension of the string.

In this case, $r = 1$, $P = 1$, $W = 1$ lb.,

$$\therefore T = \frac{4\pi^2}{g} \text{ lbs.},$$

where $\pi = 3.14159$, $g = 32.2$,

$$\therefore T = 1.226 \text{ lbs. nearly.}$$

Ex. 2. A string 3 feet long is capable of supporting a weight of 6 lbs.; how many revolutions per minute must be made by a weight of 1 lb. which is attached to it, in order that the string may break.

Taking the formula

$$\frac{T}{W} = \frac{4\pi^2 r}{P^2 g},$$

we must have in this case, $W = 1$, $T = 6$, $r = 3$; also let x be the number of revolutions required, then $x P = 60$,

$$\therefore \frac{4\pi^2 \times 3x^2}{60^2} = 6,$$

$$\text{or } x = \frac{60}{\pi\sqrt{2}} = 13.505 \text{ nearly.}$$

Hence the string will sustain 13 revolutions per minute, but 14 will break the string.

Ex. 3. If the tension of the string be 5 times the weight of the revolving body and the length of the string be 2 feet, find the velocity.

We have $$V^2 = \frac{T}{W} \times g r,$$
$$= 10 \times 32.2 = 322;$$
$$\therefore\ V = 18 \text{ feet per second, nearly.}$$

Ex. 4. When the tension is equal to the weight, the velocity is that acquired by a body in falling through a distance equal to half the radius, or the velocity is that *due to* half the radius. More generally, suppose that the tension is equal to n times the weight, then we have

$$V^2 = n g r = 2 g \times \frac{n r}{2},$$

or the velocity is that acquired in falling through the distance $\frac{n r}{2}$.

9. In the preceding investigation there is one point which requires explanation; we have supposed that there is a destruction of momentum by impact at the end of each side of the polygon upon which we have imagined the body to move, and that the sum of all the momenta so destroyed is the measure of the pressure upon the circle when the polygon becomes a circle by the indefinite increase of the number of its sides; and nevertheless we have supposed that the velocity of the body is not altered by the impacts. These two suppositions appear to be inconsistent with each other, and the apparent inconsistency requires explanation.

The difficulty arises entirely from our peculiar mode of treating the problem; it is manifest that in the actual case of motion in a circle the velocity does remain unaltered, because there is no force acting upon the body in the direction of the tangent, and it will be only necessary for us to shew that the velocity destroyed in our supposed case is such as to be insensible when the polygon becomes a circle. This we shall do by strict mathematical calculation.

Let V be the velocity upon the first side of the polygon as before; then when the body begins to move upon the second side the velocity $V \sin \frac{2\pi}{n}$ is destroyed as we have

seen, and the velocity with which the body proceeds upon the second side is $V \cos \frac{2\pi}{n}$. In like manner the velocity upon the next side is $V \left(\cos \frac{2\pi}{n}\right)^2$, and so on: hence if we suppose the body to make a complete revolution, the velocity with which it will commence a second revolution will be $V \left(\cos \frac{2\pi}{n}\right)^n$. The question is what will be the value of this quantity when n is indefinitely great.

Let a be the length of a side of the polygon, r the radius of the circumscribed circle;

$$\text{then } \frac{a}{r} = \text{chd. } \frac{2\pi}{n} = 2 \sin \frac{\pi}{n};$$

$$\text{also } \cos \frac{2\pi}{n} = 1 - 2\sin^2 \frac{\pi}{n} = 1 - \frac{a^2}{2r^2}.$$

$$\therefore\ V\left(\cos \frac{2\pi}{n}\right)^n = V\left(1 - \frac{a^2}{2r^2}\right)^n = V\left\{1 - \frac{na^2}{2r^2} + \frac{n(n-1)}{1\,.\,2}\left(\frac{a^2}{2r^2}\right)^2 + \dots\right\}.$$

Now when the polygon becomes a circle,

$$na = \text{the circumference of the circle} = 2\pi r,$$

$$\therefore\ \frac{a^2}{2r^2} = \frac{2\pi^2}{n^2},$$

$$\therefore\ V\left(\cos \frac{2\pi}{n}\right)^n = V\left\{1 - \frac{2\pi^2}{n} + \frac{\frac{1}{n^2}\left(\frac{1}{n} - \frac{1}{n^2}\right)}{1\,.\,2}(2\pi^2)^2 - \&c.\right\}$$

$$= V \text{ when } n \text{ is indefinitely great.}$$

Hence it appears that although the velocity upon the polygon is diminished by the impact, yet the quantity by which it is diminished is of such a kind that it becomes evanescent when the polygon becomes a circle. And thus the apparent difficulty is explained.

10. The case which we have now considered is of interest as being a first introduction to the theory of the

motion of the heavenly bodies; for instance, the moon very nearly describes a circle round the earth, and the earth very nearly a circle round the sun. Suppose that as a first approximation we consider the orbit of the earth about the sun to be a circle described uniformly, then the preceding investigation teaches us the force which must act upon the earth; it must be a force tending towards the sun and measured by $\frac{4\pi^2 Mr}{P^2}$, where M is the earth's mass, r its distance from the sun expressed in feet, and P the length of a year expressed in seconds. It will be observed that no force is required to urge the body to move in its orbit, but only a force tending towards the centre of the circle, that is towards the sun, which changes the direction of the earth's motion without altering its velocity. Hence the existence of this attractive force of the sun upon the earth is no assumption, but is a demonstrated fact, provided it be allowed that the earth moves uniformly in a circle which is nearly but not accurately true: when one body describes a circle about another in the centre of the circle the attraction of that central body corresponds to the pressure upon the tube, or the tension of the string in the cases before considered.

The force of the earth upon the moon on the hypothesis of circular motion is interesting because we can compare it with the force of the earth upon a body at its surface, and the comparison will help us to determine the law according to which the intensity of the earth's force changes with the distance. Now the accelerating force of the earth upon the moon determined upon the preceding principles is $\frac{4\pi^2 r}{P^2}$, where r is the moon's distance from the earth, and P the time of a revolution, or the length of one lunar month; the force of the earth's attraction at the earth's surface is g, and Newton was led to imagine that the intensity of this attraction upon any particle is less as the square of its distance from the earth's centre

is greater; we shall not now concern ourselves with the manner in which he might have been led to this law, but we will test it in the case of the moon, so far as that can be done upon the hypothesis of the moon's orbit being a circle. Let R be the radius of the earth considered as a sphere, then if the Newtonian law be true we must have

$$\frac{4\pi^2 r}{P^2} : g :: \frac{1}{r^2} : \frac{1}{R^2},$$

$$\text{or } g = \frac{4\pi^2 r^3}{P^2 R^2};$$

we shall now proceed to calculate the value of the right-hand side of this equation, making use of the following data,

$R = 20888700$ feet,

$r = R \times 59.9643$,

$P = 27.3217$ days;

$$\therefore g = \frac{4\pi^2 (59.9643)^3 \times 20888700}{(27.3217 \times 24 \times 60 \times 60)^2} = \frac{\pi^2 (59.9643)^3 \times 208887}{(273.217 \times 432)^2}.$$

From the tables we have the following,

$$\log \pi = .4971495; \therefore \log \pi^2 = .9942990$$
$$\log 59.9643 = 1.7778928; \therefore \log (59.9643)^3 = 5.3336784$$
$$\log 208887 = 5.3199115$$
$$\overline{11.6478889}$$
$$\log 273.217 = 2.4365077; \therefore \log (273.217)^2 = 4.8730154$$
$$\log 432 = 2.6354837; \therefore \log (432)^2 = 5.2709674$$
$$\overline{10.1439828}$$
$$\therefore \log g = 1.5039061,$$
$$\text{or } g = 31.9 \text{ nearly.}$$

We have before mentioned that the value of g as determined from terrestrial observations is 32.2, which differs

from the above by only .3 of a foot; a quantity not to be neglected, but sufficiently small to render the law of attraction highly probable when it is considered that the whole calculation has been conducted upon an hypothesis which is only approximately true.

11. The formula for the pressure upon the circular tube, in which we have supposed a particle to move, has been investigated on the supposition of the velocity being uniform; it is not however difficult to shew that the same formula would hold true if the velocity were not uniform, that is, if the body were subject to the action of a force in the direction of its motion. For suppose the body were to move uniformly through half the circumference, and then to receive an impulse which increased its velocity through the other half; then it is clear that the formula would hold for each half circumference, only that in the second V will have a different value from that which it has in the first; the same thing will hold true if we suppose that the body receives an impulse at the end of each quadrant; and so if we conceive the circumference to be divided into an infinitely great number of small arcs, and suppose the body to receive a small impulse at the end of each, the formula will still be true, but V will have a different value for each indefinitely small arc: but in this way we may arrive at the case of a particle moving in any manner whatever, or under the action of any force, and the conclusion at which we have arrived will still be true, and hence it appears that if a body move in a circular tube under the action of a force in the direction of its motion, and if V be the velocity at any given instant, the pressure upon the tube at that instant will be $\frac{MV^2}{r}$.

12. And hence we may arrive at the still more general case of a body moving in a circular tube under the action of any forces whatever; for let the forces be resolved into two portions, T in the direction of the motion, and

N in the direction perpendicular to the motion, that is normal to the circle. Then N will have no effect upon the velocity, but will produce only pressure upon the curve; hence omitting all consideration of the motion the pressure would be MN, (N being reckoned as an accelerating force); and we have before seen that a pressure measured by $\frac{MV^2}{r}$ is due to the motion, hence taking both sources of pressure into account, the pressure upon the tube will be $MN \pm \frac{MV^2}{r}$, the upper or lower sign being taken according as the motion tends to increase or decrease the pressure due to the normal force.

13. We will now consider the case of a body moving down the surface of a curve, or in the interior of a tube, in a vertical plane, under the action of gravity only; and we are able to prove the following general proposition.

PROP. *The velocity acquired by a body in falling down a curve in a vertical plane is that which would have been acquired by the body in falling freely through the same vertical height.*

For instance, let a body fall from a point A upon such a curve and reach another point B, and let the point B be vertically lower than A by the distance h, then the velocity of the body at B will be $\sqrt{2gh}$.

This proposition we shall prove by supposing the body in the first instance to fall down a succession of inclined planes; now we have already seen (Art. 27, p. 41) that in falling down an inclined plane the velocity is that acquired in falling through the vertical height of the plane, hence at the bottom of the first plane the proposition is true, and it will be true for the bottom of the second plane *if there be no velocity destroyed in passing from one plane to the other*; and so of the third, fourth, &c. Hence if a body fall down any number of inclined planes, the velocity

G 2

acquired will be that due to the vertical height, *provided no velocity be destroyed in passing from one plane to another.* But velocity *is* lost in passing from one plane to another by impact, and therefore in the case of a number of planes the proposition is not true; but the smaller the angle between two consecutive planes, the smaller will be the amount of velocity destroyed, and when the system of planes becomes such as to be not capable of being distinguished from a continuous curve we are justified in assuming that the quantity of velocity destroyed is imperceptible. That this is so may be seen in two ways; in the first place we have shewn that such is the case in an analogous problem, (Art. 9, p. 143) and the principles of that demonstration apply to this, although we are not able with elementary mathematical methods to work out the result: and in the next place, if any velocity were destroyed, the destruction must be due to the action of the curve; now the action of the curve is always exactly perpendicular to the motion, and therefore has no effect in retarding it, and hence the supposition of velocity being destroyed involves us in an absurdity.

On the whole, therefore, we conclude that the proposition which is true for an inclined plane is true for any curve; that is, the velocity acquired in falling down it is that due to the vertical height.

14. Let us apply the preceding proposition to the case of a particle moving upon a circle in a vertical plane.

Ex. 1. A particle runs down the interior of a vertical semicircle; to find the pressure at the lowest point.

Let the radius be r; then if V be the velocity at the lowest point, we have

$$V^2 = 2gr.$$

The pressure due to the weight is Mg, hence the whole pressure will be (Art. 12)

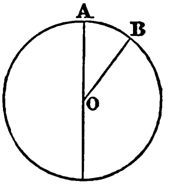
$$Mg + \frac{MV^2}{r} = Mg + 2Mg = 3Mg.$$

Or the pressure is three times the weight, as announced by anticipation in Art. 3, p. 138.

Ex. 2. A particle runs down the exterior surface of a vertical circle, to find the point at which it will leave the curve.

This is a case in which the motion tends to lessen the pressure due to the direct action of gravity; and as the pressure due to the motion increases and that due to the weight diminishes, the particle soon arrives at a point for which they are equal; at this point it leaves the curve and describes a parabola.

Let A be the highest point of the circle, B any other point, O the centre, $AOB = \theta$, $AO = r$.



Then the vertical distance of B from A is $r - r\cos\theta$, and therefore if V be the velocity at B, we have

$$V^2 = 2gr(1 - \cos\theta),$$

$\therefore$ the pressure due to the motion

$$= \frac{MV^2}{r} = 2Mg(1 - \cos\theta).$$

Again, the resolved part of the weight Mg in the direction BO is $Mg\cos\theta$, and therefore the actual pressure at B will be

$$Mg\cos\theta - 2Mg(1 - \cos\theta) = Mg(3\cos\theta - 2).$$

Hence the particle will leave the curve, when

$$3\cos\theta - 2 = 0,$$

$$\text{or } \cos\theta = \frac{2}{3}, \text{ or } \theta = 48°48' \text{ nearly};$$

in other words the particle will leave when it has passed over a vertical distance equal to one-third of the radius.

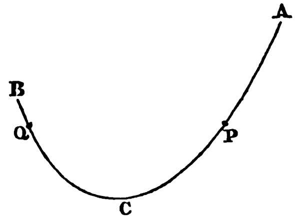
Ex. 3. To find the latus rectum of the parabola which the particle describes after leaving the circle.

The direction of projection is that perpendicular to OB, if B be the point at which the body leaves the curve, and therefore the value of θ above determined will be the angle which the direction of projection makes with the horizontal line through B; hence, referring to the general expression for the latus rectum given in Art. 12, p. 67, we have

$$\text{latus rectum} = 4r(1 - \cos\theta)\cos^2\theta$$

$$= \frac{4}{3} \times \frac{4}{9} r = \frac{16}{27} r.$$

15. If we have a particle moving upon the interior of any vertical curve ACB, of which C is the lowest point, and if we allow the particle to fall from a point P, it will descend to C and ascend again on the other side of C to a point Q, such that the vertical height of Q above C is equal to that of P, in other words Q and P lie in the same horizontal line. This is clear from the fact that the velocity at C is that which would be acquired in falling through the vertical height of P above C, and the velocity so acquired will be destroyed when the body has moved upwards through a vertical space equal to that through which it has fallen. From Q the body will descend, and passing C will rise to P, and so on perpetually.



16. If we suppose the curve to be a circle, the motion of the particle will be precisely the same as if it were attached by a fine thread to a fixed point at the centre of the circle; and we are thus brought to a case of constrained motion which is of the utmost practical importance.

A heavy body suspended by a thread is called a *pendulum*; practically of course the body must have appreciable magnitude, and the thread must have perceptible weight; we shall however be concerned only with the extreme mathematical case of a *particle* suspended by an *indefinitely fine* thread, or moving upon a smooth inverted semicircle.

A particle so suspended will oscillate through equal angles upon opposite sides of the vertical line through the point of suspension, that is, of the position which the pendulum would occupy if hanging at rest. The problem is to determine the time in which the pendulum will make an oscillation, and this is a problem of first-rate importance in physical science, as will be seen hereafter.

The complete determination of the time of oscillation in this general case will however be beyond our powers; it will be easily understood that the time of oscillation

depends upon the extent of the arc through which the pendulum vibrates, and the law according to which the time depends upon the arc of vibration is very complicated. Mathematicians have however discovered that there is a certain curve, upon which if the particle be made to oscillate the time of oscillation is *not* dependent upon the extent of the arc of vibration; on this account the determination of the time of oscillation upon this curve will be found much more easily than in the case of a circle, and we shall therefore first consider this more simple case.

17. The curve in question is called a *cycloid;* it is the curve described by any point in a carriage-wheel when the wheel rolls upon a plane; or we may give the following definition:

DEF. *When a circle rolls upon a fixed straight line, the locus of any point in the circumference is called a* cycloid.

The form of the curve is shewn in the accompanying

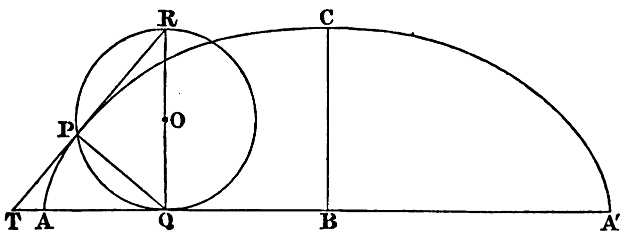


figure. O is the centre of the rolling circle in any position, P the point which traces out the cycloid, AA' the straight line upon which it rolls, and which is called the base or directrix. The describing point P will rise from the point A in the directrix from which it starts until it reaches the highest point C at a distance BC from the directrix equal to the diameter of the rolling circle; it then again approaches the directrix, describing an arc CA' precisely similar to the arc CA; BC is called the axis of the cycloid.

We must now digress from the immediate subject of this Chapter for the purpose of investigating some of the chief properties of the cycloid.

18. Prop. *To draw a tangent to a cycloid.*

In order to do this we shall consider the tangent at any point of the cycloid as the direction in which the generating point is moving. This may in fact be regarded as the simplest mode of defining the tangent of a curve which is generated by a point moving according to a certain law; the tangent to a circle, for example, though otherwise defined by Euclid, admits of such definition; that is, if we regard a circle as generated by a point which moves under the condition of being always at a given distance from a certain fixed point, then the direction of the motion of the generating point is constantly changing, and if at any instant we suppose the point to continue to move in the direction in which it is moving at that instant, then the straight line in which the generating point moves is the tangent to the circle at the point which is being traced at the instant in question.

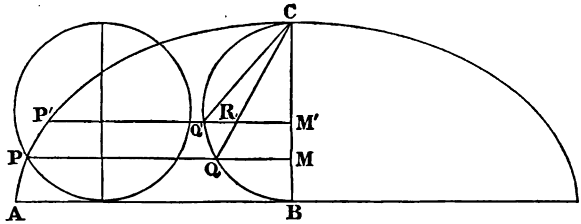
Now let us apply this notion of a tangent to the case of the cycloid. Let P be the generating point, and Q the point of the circle in contact with the directrix at any given instant, (see figure of preceding Article). Join PQ; then since the generating circle *rolls* upon the directrix, the point Q may be regarded as for an instant at rest, and therefore the motion of P will be for a *very short time* like that of a point describing a circle round Q, in other words, its motion is perpendicular to PQ, and therefore the tangent of the cycloid at P is perpendicular to PQ.

Hence we have a very simple construction for the tangent of the cycloid. Draw the diameter QR of the generating circle from the point of contact Q; join PR, which will be the tangent required.

19. Prop. *To find the length of the arc of a cycloid.*

Let P, P' be two contiguous points in a cycloid; on the axis BC describe a semicircle, and through P, P' draw straight lines PQM, $P'Q'M'$, perpendicular to BC and

cutting the semicircle in Q, Q' respectively. Join CQ, CQ' and let R be the intersection of CQ, $P'M'$.



We shall now prove that if P and P' be *indefinitely* near together, the small arc of the cycloid PP' will be *indefinitely* nearly equal to twice the difference between the two chords CQ and CQ'.

When P' is indefinitely near to P, the small portion of the arc of the cycloid PP' may be regarded as being the same as the straight line PP', and this straight line may be regarded as coinciding with the tangent at P; but CQ, being by the preceding proposition parallel to the tangent at P, is parallel to PP', hence $PP' = QR$, P and P' (it will be remembered) being supposed to be indefinitely near to each other.

Again, let a be the radius of the generating circle, $BCQ = \theta$, $QCQ' = \alpha$, then

$$PP' = QR = 2a\cos\theta - 2a\cos^2(\theta+\alpha)\sec\theta$$
$$= 2a\sec\theta\left\{\cos^2\theta - \cos^2(\theta+\alpha)\right\},$$
$$CQ - CQ' = 2a\cos\theta - 2a\cos(\theta+\alpha);$$
$$\therefore \frac{PP'}{CQ - CQ'} = \sec\theta\left\{\cos\theta + \cos(\theta+\alpha)\right\} = 2 \text{ when } \alpha = 0;$$

that is, if P and P' be indefinitely near together,

$$PP' = 2(CQ - CQ');$$

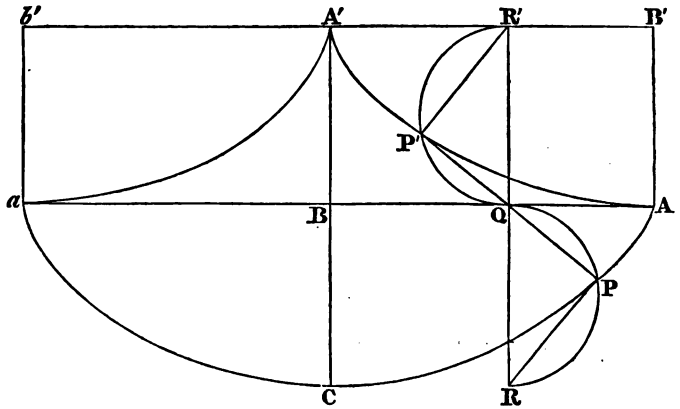
and as this is true for every indefinitely small portion of the arc of the cycloid, and as CQ is equal to the chord of the generating circle which touches the cycloid at P, we conclude that *the arc of the cycloid measured from the vertex to any point is equal to twice the chord of the gene-*

rating circle which touches the curve at that point. In the figure of Art. 17, $CP = 2PR$.

20. We shall now shew how a particle may be made to oscillate in a cycloid; theoretically we may suppose a groove to be cut in the form of a cycloid, and a particle to be allowed to oscillate in this groove, but we shall be able to prove that the cycloid possesses a property which enables us to make a particle suspended by a thread oscillate in such a manner as to trace out a cycloidal path, and the motion of a particle so suspended will be precisely the same in a mechanical point of view as if it were constrained by a groove.

21. PROP. *To make a pendulum oscillate in a given cycloid.*

Let APC be a given semicycloid, having directrix AB and axis BC; produce CB to A', making $BA' = BC$, and complete the rectangle $A'BAB'$: with $A'B'$ as directrix and AB' as axis, describe the semicycloid $AP'A'$.



From any point R' in $A'B'$ draw $R'QR$ equal and parallel to $A'BC$, and cutting AB in Q; on RQ, $R'Q$ describe the two generating semicircles QPR, $QP'R'$; join QP, PR, QP', $P'R'$.

Then the circular arc $QP = AQ$, as is manifest from

the manner in which the cycloid is generated; and in like manner, arc $QPR = AB$;

$$\therefore \text{ arc } PR = BQ = A'R' = \text{arc } P'R';$$
$$\therefore PR = P'R',$$
$$\text{also } QR = QR',$$

and each of the angles QPR, $QP'R'$ is a right angle;

$\therefore$ the triangles QPR, $QP'R'$ are equal in all respects.

Hence angle $PQR = P'QR'$, and therefore PQP' is a straight line.

Also PP', which is a tangent to $A'P'A$ at $P' = 2P'Q$ = arc $P'A$ by Art. 19.

Hence if a string of length $A'P'A$, fixed at A' and wrapped upon the semicycloid $A'P'A$ be unwrapped, beginning at A, a particle attached to its extremity will trace out the semicycloid APC. And by means of another semicycloid $A'a$, the particle may be made to describe the other half of the cycloid ACa.

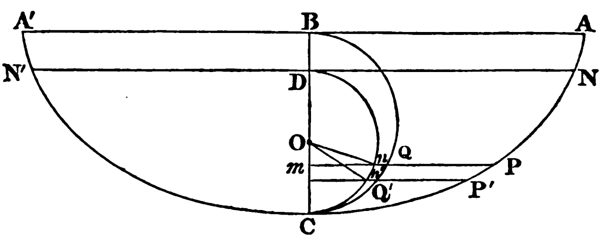
22. We now proceed to determine the time in which a particle will make an oscillation when constrained to move in a vertical cycloidal groove, or (which is the same thing), when constrained to trace out a cycloid in the manner explained in the preceding Article. The general problem of finding the time of oscillation, when a particle is constrained to move upon a curve, is beyond the reach of the mathematics employed in this book, but the solution can be effected in the particular case of the cycloid. The principle of the method which we shall adopt is as follows; we shall suppose the arc of the cycloid to be divided into an indefinite number of small portions; we shall know the velocity which the particle has at the commencement of each of these portions by Art. 13, and we shall suppose the particle to describe each small arc uniformly with the velocity which it has at its commencement; the time which the particle takes to describe the cycloid upon this hypothesis will not be precisely that which is actually required, but it will be more and more nearly so as the portions into

which the arc is divided are smaller, and if we find the time on the supposition of the portions being *infinitely* small, the time so found will be that actually required by the particle for a true cycloidal oscillation. The student will perceive that this method is precisely the same in principle as that adopted in Art. 12, p. 28.

Obs. The student is supposed to be acquainted with the proposition, proved in all treatises upon Trigonometry, that when θ is indefinitely small $\frac{\sin\theta}{\theta} = 1$, θ being expressed in the circular measure.

23. Prop. *To find the time of oscillation of a heavy particle moving on the surface of a cycloid.*

Let ACA' be the cycloid having its axis BC vertical, N a point from which a heavy particle is allowed to de-



scend; then if we draw NDN' horizontal, N' will be the point to which the particle will ascend. On BC describe the semicircle BQC, and on DC the semicircle DnC having O for its centre.

Let P be the position of the particle at any given time, P' its place at an indefinitely short time after it has passed P; through P, P' draw the horizontal lines $PQnm$, $P'Q'n'$, cutting the circles above described in Q, Q' and n, n' respectively. And join On, On'.

Let the angles $COn' = a$, $nOn' = \theta$, $OC = r$, $BC = 2a$. Then the velocity of the particle at $P = \sqrt{2g \,.\, Dm}$ (Art. 13, p. 147.)

$\therefore$ the time of describing PP' uniformly with the velocity which the particle has at $P = \dfrac{PP'}{\sqrt{2g.Dm}}$.

But $PP' = 2(CQ - CQ')$, Art. 19, and if we suppose BQ, CQ to be joined, we have from similar triangles BCQ, QCm,

$$BC : CQ :: CQ : Cm;$$

$$\therefore\ CQ^2 = BC \,.\, Cm,$$

$$\text{in like manner, } Cn^2 = DC \,.\, Cm;$$

$$\therefore\ \frac{CQ^2}{Cn^2} = \frac{BC}{DC} = \frac{a}{r};$$

$$\text{and } CQ = \sqrt{\frac{a}{r}}\, Cn = \sqrt{ar}\,\text{chd}\,(\alpha + \theta) = 2\sqrt{ar}\sin\frac{\alpha + \theta}{2};$$

$$\text{similarly } CQ' = 2\sqrt{ar}\sin\frac{\alpha}{2};$$

$$\text{also } Dm = r + r\cos(\alpha + \theta) = 2r\cos^2\frac{\alpha + \theta}{2};$$

$$\therefore\ \text{time of describing } PP' = \frac{4\sqrt{ar}\left(\sin\dfrac{\alpha+\theta}{2} - \sin\dfrac{\alpha}{2}\right)}{\sqrt{4gr}\cos\dfrac{\alpha+\theta}{2}}$$

$$= 4\sqrt{\frac{a}{g}}\,\frac{\cos\left(\dfrac{\alpha}{2} + \dfrac{\theta}{4}\right)\sin\dfrac{\theta}{4}}{\cos\left(\dfrac{\alpha}{2} + \dfrac{\theta}{2}\right)}$$

$= \sqrt{\dfrac{a}{g}}\,\theta$, when θ is indefinitely small, since in that case $4\sin\dfrac{\theta}{4} = \theta$, (see last Art.), and $\cos\left(\dfrac{\alpha}{2} + \dfrac{\theta}{4}\right)$ and $\cos\left(\dfrac{\alpha}{2} + \dfrac{\theta}{2}\right)$ each become equal to $\cos\dfrac{\alpha}{2}$.

It will be seen that all the indefinitely small angles, such as θ, corresponding to the small arcs PP' between N and N', will together make up four right angles; hence the time of describing the arc NCN' will be $2\pi\sqrt{\frac{a}{g}}$; and the time of a complete oscillation, that is, of moving from N to N' and back again from N' to N will be $4\pi\sqrt{\frac{a}{g}}$.

24. The time of oscillation it will be observed depends solely upon the values of a and g, and is entirely independent of the point in the cycloid from which the particle starts. Hence the time of vibration in a cycloid is said to be independent of the arc of oscillation, and this curious property of the curve is expressed by saying that the curve is isochronous.

25. If l be the length of a string by which a particle is suspended, and if the particle be made to perform cycloidal oscillations in the manner above explained, it will be seen that $l = 4a$, and therefore the time of semivibration will be $\pi\sqrt{\frac{l}{g}}$.

26. If we suspend a particle by a string of length l, and cause it to oscillate without using the artifice necessary to make it trace out a cycloid, the particle will move in a circular arc and the time of oscillation just investigated will not be correct; in fact, the time of oscillation will depend upon the extent of the arc of vibration, and will not depend entirely upon l and g as in the case of the cycloid; it is manifest however that the smaller the arc of vibration the smaller will be the difference between the time of oscillation on the supposition of the particle moving in a cycloid, and the time on the supposition of the particle moving in a circle, and hence if we suppose the excursions of the pendulum to be *very* small we may say without

sensible error that the time of a semivibration will be $\pi\sqrt{\dfrac{l}{g}}$.

27. The pendulum here described is, however, purely theoretical. Our investigation has been conducted upon the hypothesis of a *particle* being attached to a string *without weight;* now no portion of matter capable of being used as a pendulum can be regarded accurately as a particle, and the string used, however fine, must have weight; hence no pendulum can be constructed such as that which we have described. This theoretical pendulum consisting of a single particle and a string without weight is called a *simple* pendulum, and since no such pendulum can be actually constructed it is necessary to determine by calculation from observations of such a pendulum as we can actually construct, what would be the length of the simple pendulum. This problem has been solved in several ways; it is not within the compass of this treatise however to give the solution, it must be sufficient to state that from observations made with a properly constructed pendulum it is possible to deduce the length of a simple pendulum, which would oscillate in precisely the same time as the actual pendulum upon which the observations are made.

28. One of the most important applications of the pendulum is the determination of the value of the accelerating force of gravity. It may be determined by observation of falling bodies, as explained in p. 102, but no method is so accurate as that which depends upon the observation of the time of oscillation of a pendulum, because by taking the result obtained from a large number of oscillations, the errors of observation may be diminished to almost any extent.

29. PROP. *To determine from observations of the pendulum the value of the accelerating force of gravity.*

By comparing the time of oscillation of a pendulum with the beats of the pendulum of a good clock, the length of the simple pendulum which makes a semivibration

in one second may be determined. This can be done with great accuracy; let L be its length expressed in feet: then the unit of time being one second, we have

$$\pi\sqrt{\frac{L}{g}} = 1,$$

$$\therefore \quad g = \pi^2 L.$$

Hence if L, or the length of a *seconds'* pendulum, be known, g is also known; and as L can be determined with great nicety, g can also be determined with equal accuracy.

When observations of the length of the seconds' pendulum are made at different parts of the earth's surface, it is found that the length varies between small limits; this is due partly to the fact of the earth not being truly spherical but rather flattened at the poles, and partly to the effect of centrifugal force being greatest at the equator and diminishing as we go north or south; the length of the seconds' pendulum at the equator is about 39 inches, at the poles $39\frac{1}{5}$; in the latitude of London the length is about $39\frac{1}{8}$.

30. Prop. *To find the number of seconds which a pendulum will lose in a day, when lengthened by a given small quantity, supposing the pendulum to be previously a seconds' pendulum.*

Let a be the additional length, and T the time of a semi-oscillation, x the number of seconds lost in 24 hours.

$$\text{Then } T = \pi\sqrt{\frac{L+a}{g}} = \pi\sqrt{\frac{L}{g}}\left(1+\frac{a}{L}\right)^{\frac{1}{2}}$$

$$= \pi\sqrt{\frac{L}{g}}\left(1+\frac{a}{2L}\right) \text{ nearly, (if we expand by}$$

the Binomial Theorem and retain only the first two terms,)

$$= 1 + \frac{a}{2L}, \text{ since by hypothesis } \pi\sqrt{\frac{L}{g}} = 1;$$

$$\therefore \; x = 24 \times 60 \times 60 - \frac{24 \times 60 \times 60}{T} = 24 \times 60 \times 60 \times \frac{a}{2L} \text{ nearly.}$$

Suppose, for example, that $\frac{a}{L} = \frac{1}{100}$, then the number of seconds lost $= \frac{24 \times 60 \times 60}{200} = 432.$

31. Prop. *To ascertain the height of a mountain by observation of the pendulum.*

To solve this problem we shall assume that the intensity of the force of gravity varies inversely as the square of the distance of the point at which the observation is made from the centre of the earth, and the earth we shall suppose to be a sphere of 4000 miles radius, which though not strictly true is sufficiently near the truth for our present purpose.

Suppose then the height of a mountain to be x feet, and let g' be the accelerating force of gravity at its summit, g at its base; then

$$g' = g\left(\frac{4000 \times 1760 \times 3}{4000 \times 1760 \times 3 + x}\right)^2.$$

Hence if L be the length of the seconds' pendulum at the base of the mountain, the time of oscillation of such a pendulum at the summit

$$= \pi\sqrt{\frac{L}{g'}} = \pi\sqrt{\frac{L}{g}}\,\frac{4000 \times 1760 \times 3 + x}{4000 \times 1760 \times 3}$$

$$= 1 + \frac{x}{4000 \times 1760 \times 3}, \text{ since } \pi\sqrt{\frac{L}{g}} = 1 \text{ by hypothesis.}$$

Now suppose it to be ascertained by observation that the pendulum at the summit of the mountain loses n oscillations in 24 hours, then the time of oscillation will be

$$= \frac{24 \times 60 \times 60}{24 \times 60 \times 60 - n} = 1 + \frac{n}{24 \times 60 \times 60} \text{ nearly;}$$

$$\therefore x = n\,\frac{4000 \times 1760 \times 3}{24 \times 60 \times 60} = n \times 245 \text{ nearly.}$$

Suppose, for example, that $n = 10$, then the height of the mountain is 2450 feet.

CONVERSATION UPON THE PRECEDING CHAPTER.

T. Do you clearly understand the statements made in this Chapter respecting *Centrifugal Force?*

P. It seems that there is in reality no such force; that is to say, what we call centrifugal force is not a force.

T. It is not a force due to the action of some cause external to the body, and which like gravity would act upon the body precisely in the same manner whether it be at rest or in motion, but it is an imaginary force capable of producing a certain effect, which effect is due to the motion of the body. Thus if you make a stone to revolve at the extremity of a string, the string undergoes a tension which is due entirely to the stone's motion, and we should say that the string is stretched by the centrifugal force of the stone, but more properly it would be stated that the tension of the string is due to the continual checking of the tendency which the stone has to move in a straight line instead of a circle. Perhaps the less you use the term the better, as without great care it is likely to give rise to erroneous notions.

P. I found the other day in a scientific dictionary explanations of the terms Centrifugal and Centripetal Force, which, I believe, did convey a mistaken notion to my mind. They were as follows:

Centrifugal Force is that by which a body revolving about a centre, or about another body, endeavours to recede from it.

Centripetal Force is that by which a moving body is perpetually urged towards a centre, and made to revolve in a curve, instead of a right line.

T. It would be incorrect to suppose, as perhaps you might from reading these definitions, that the centrifugal and centripetal forces under the action of which bodies

are supposed to describe curvilinear paths were forces of the same kind, one tending towards a centre, the other from the centre. You will observe, however, that in the definition of Centrifugal Force the term *endeavours* is used, implying that the force is due to the body's own motion, not to any external influence; no such term is applied to Centripetal Force; so that the definitions are correct, though it must be allowed that the affinity of the names Centripetal and Centrifugal is rather likely to engender the notion that they are forces of the same kind but opposite in direction.

P. Is it not sometimes said that the motion of the earth round the sun is maintained by the equilibrium of the centripetal and centrifugal forces?

T. Such an expression may be used, and the meaning of it is this. The sun exerts a powerful attraction upon the earth; we cannot say what the cause of the attraction is, at present the knowledge of its existence and of the laws according to which it acts form the limits of our knowledge concerning it. The effect of this attractive force, if the sun and earth were placed at a distance from each other, and both at rest, would be to cause the earth to approach the sun, or more strictly, to cause the sun and earth to approach each other, and the earth falling into the sun would cease to exist as a separate body. But the earth instead of being thus placed at rest has been projected in a direction nearly perpendicular to the line joining it with the sun; if the earth had been projected in an exactly perpendicular direction, and with a proper velocity, the earth would have moved uniformly in a circle round the sun; as it is, the motion is nearly circular and nearly uniform, taking place in an ellipse of small excentricity having the sun in one of the foci. When the earth is at its smallest distance from the sun, or in perihelion, its velocity is too great to allow it to describe a circle, consequently its distance from the sun increases, until at

length the sun's attraction causes the distance again to diminish, and so on perpetually. If the motion of the earth were circular, no portion of the sun's attraction would be employed in accelerating and retarding the earth's motion, all the force exercised by the sun would be employed in checking the tendency of the earth to move out of the circle, in other words, in destroying the effect of centrifugal force; in this case then it would be said that there is equilibrium between the centripetal and centrifugal forces; in the actual case of the earth it would be more correct to say that the greater part of the centripetal force is expended in counteracting the centrifugal force, and that a small part is employed in accelerating and retarding the motion.

P. You said that the earth would describe a circle if projected with a proper velocity; how could that velocity be ascertained?

T. It appears at once from the investigation of Art. 6, p. 140. For the result of that investigation was to shew that when a particle revolves uniformly with a velocity V, in a circular tube, the pressure upon the tube is measured by $\frac{MV^2}{r}$, or the corresponding accelerating force is $\frac{V^2}{r}$; suppose that instead of a tube we have an attractive force tending to a certain point, and at the distance r from this centre let f be the accelerating force which the attraction can exert, in other words, let f be the velocity which the attractive force of the centre could generate in one second in a body whose mass is unity if the force were to act uniformly during that second. Now if we project a particle in a direction perpendicular to the line joining it with the centre of force, and with a velocity such that

$$V^2 = fr, \dots\dots\dots\dots\dots\dots\dots (A)$$

the particle will be under precisely the same circumstances as when projected within a circular tube with a velocity

V; consequently the particle will continue to describe the circle, and the velocity will be throughout the motion that given by the equation (A).

P. If then the radius of the circle and the intensity of the attractive force of the centre be given, it will always be possible to project a body with such a velocity as to make it move in a circle.

T. Yes; if we know the velocity we can find the force as in Art. 10, and if we know the force we can determine the velocity. Hence you will perceive that the time required by a stone to fall from my hand to the earth, and the time required by the moon to revolve round the earth, that is, the length of the month, are quantities intimately connected.

P. I am much struck by the manner in which science connects facts apparently unconnected; I suppose that it was somewhat in the manner given in p. 145, that Newton satisfied himself that the attraction of the earth was really the force which retained the moon in its orbit.

T. Yes; and the evidence would be still more satisfactory if we followed out the principle of universal gravitation more exactly, for you will remember that if the theory of universal gravitation be true, there are other bodies acting upon the moon besides the earth; of these bodies one is of such magnitude that its effect can be by no means neglected, I mean the sun. If the force of the sun had been taken into account as well as that of the earth in the calculation of page 145, the resulting value of g would have been nearer to the truth. Without making that calculation, which would take us too far from our present subject, I will remark that it is easy to see that the result there obtained is a little too small: for although the force of the sun upon the moon is very variable at different times, still inasmuch as the orbit of the sun (considered as moving round the earth) is always ex-

terior to that of the moon, the general or *mean* effect of the sun is to draw the moon away from the earth, or to diminish apparently the earth's attraction; now the value of g found in page 145 is the value calculated without considering that the moon's motion is in part due to the action of the sun, and is therefore smaller than it ought to be; in fact, suppose G to be the true value of the force of gravity, and suppose that the mean effect of the sun is to diminish this by a small quantity f, then the quantity g found in page 145 is $G-f$, or is rather too small. I assume the quantity f to be small, it will in fact be so, in consequence of the great distance of the sun.

But we are losing sight of the subject of the Chapter, which is constrained motion, and I will therefore only recommend to you to study the history of the discovery of the truth that the moon is held in her orbit by the same force which makes a stone fall to the ground, as you will find it in any biography of Newton; the discovery formed one of the most remarkable epochs in the history of science, and the circumstances related as connected with it, such as the abandoning of the thought by Newton when in consequence of using an erroneous value of the earth's radius he fancied that he had disproved his theory, and the intensity of his excitement when he subsequently found that his expectations were likely to be realized, give to the history a much more than ordinary interest.

P. I suppose then that we may now pass on to the cycloid and the pendulum. I feel some difficulty concerning the method given of drawing a tangent to a cycloid.

T. With regard to the general notion of a tangent to a curve at any point being the direction in which the point which traces out the curve is then moving, I think that there is no difficulty in the conception and that it is one of the best modes of considering tangents. The only question, I suppose, is to determine in the case of a rolling circle what is the direction of motion of any point in the circumference.

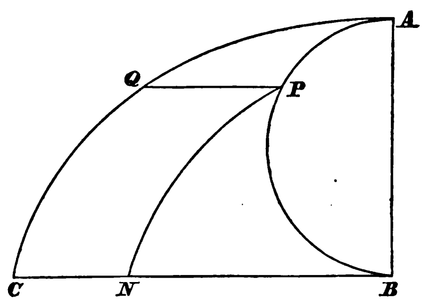
In consequence of successive portions of the circumference coming in contact with the directrix the direction of motion of any given point is continually changing, but if we regard only a very small motion of the circle it seems to me that we may safely conclude that the motion of a portion of the circular area resting upon the point of contact, the triangle *RPQ*, for instance, (figure p. 151), will move precisely as if the rest of the circle were cut away. Let us then confine our attention to the triangle *RPQ*; it is manifest that in this case the point *P* moves in a circle round *Q* or its motion is perpendicular to *PQ*, and hence, I think, you may infer that even when the triangle *RPQ* forms a portion of the rolling wheel the direction of motion of *P* is for an instant the same, that is, perpendicular to *PQ*. I admit however that the question is not without difficulty in consequence of the continual change of the point of the circle which is in contact with the directrix.

P. I must take time to consider the question.

T. In the mean time I will furnish you with another mode of treating the cycloid which does not introduce the same difficulty.

It is not hard to see that the cycloid, instead of being generated as described in page 151, may be supposed to be generated as follows.

Let *APB* be a fixed circle standing upon the straight line *BC*; in *APB* take any point *P*, and from *P* draw *PQ* parallel to *BC* and equal to the arc *AP*, then the locus of *Q* will be a cycloid *AQC* of which the axis is *AB* and the directrix *BC*. This follows at once from the fact that



if we suppose the circle to roll until *A* comes in contact

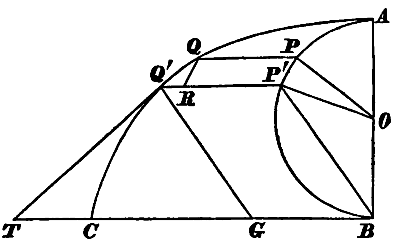
with C, and suppose N to be the point in BC, which came in contact with P, we shall have

$$BC = \text{arc } APB,$$
$$\text{and } BN = \text{arc } PB,$$
$$\therefore\ CN = \text{arc } AP;$$

but it is evident that CN must be equal to QP since the curves described by P and A are precisely the same in form and only differ from each other in the circumstance of one being shifted a little in a horizontal direction from the other, in other words the locus of P and A are two equal cycloids having the same directrix and axes distant from each other by the quantity CN.

If we adopt the method of generating the cycloid which I have now described, the property of the tangent may be easily investigated, as follows.

Let P, P' be two contiguous points in the fixed circle APB, whose centre is O, Q, Q' the corresponding points in the cycloid; join QQ' and produce it to meet the directrix in T, then when Q and Q' are indefinitely near together $TQ'Q$ will become a tangent to the cycloid. Draw QR parallel to the chord PP' and meeting $P'Q'$ in R; join PO, $P'O$, $P'B$, and draw $Q'G$ perpendicular to $TQ'Q$.



Then because $QRP'P$ is a parallelogram, $QR = PP'$,

and $P'R = PQ$.

Also $Q'R = Q'P' - P'R = Q'P' - PQ = PP'$ by the mode of generating the cycloid,

$= QR$, if we suppose the arc PP' and the chord P to be equal;

$$\therefore\ \text{angle } RQ'Q = \text{angle } RQQ',$$
$$\text{and } QQ'R = \text{half } QRP'$$
$$= \text{half } P'OA = P'BA,$$

$$\therefore\ GQ'R = GBP',$$

or $Q'G$ is parallel to $P'B$;

hence the normal to the cycloid at Q' is parallel to $P'B$, or the tangent is parallel to $P'A$, which gives the same rule for drawing the tangent as that investigated in page 152.

P. You have regarded the tangent in the demonstration in quite a different light from that in which it is viewed according to the other method.

T. I have regarded the tangent as the straight line joining two points in the curve, when those two points are indefinitely near together; this is a very good notion to have of a tangent, and indeed is the one generally adopted in the higher mathematics.

Let me call your attention to the fact that in the demonstration the small arc PP' and the chord PP' have been regarded as equal; this is justified by the proposition proved in trigonometry, namely, that the sine and the arc are ultimately equal, or which is the same thing, that the arc and the subtending chord are ultimately equal.

P. Is the method of making a pendulum move in a cycloid, given in p. 154, of any practical utility?

T. Very little, if any; sometimes it is attempted to make the oscillations of a pendulum isochronous by such means, but the result can generally be sufficiently nearly attained by making the excursions of the pendulum very small. In fact, the difficulty of dealing with the pendulum practically is of quite another kind. Treating the pendulum as a regulator of clocks a difficulty arises from the change in the form of the pendulum in consequence of change of temperature; the irregularity arising from this cause is remedied in several ways, but I must refer you upon this subject, as well as upon that of *Escapements*, to treatises on clockmaking, reminding you that the complete investigation of the subject involves dynamical considerations upon which we have not yet entered. Treating the pendulum as a scientific instrument the great problem is to deduce in the

most accurate manner from an actual pendulum the length of a theoretical or simple pendulum, that is, a pendulum consisting of a single particle suspended by a thread, which would vibrate in the same time as the actual pendulum. In a mathematical point of view this problem presents no difficulty, but it is by no means easy to obtain accurate results in practice; very good results have however been obtained by several different methods, and the problem may be regarded as completely solved.

P. It is stated in page 160, that when observations are made of the length of the seconds' pendulum at different parts of the earth's surface it is found that the length varies between small limits, and that this variation is due partly to the earth's form and partly to centrifrugal force: I am not sure that I understand this.

T. If the earth were perfectly spherical and homogeneous, there would be no reason why the attraction should be different at different points of its surface. But the earth is found not to be perfectly spherical; it may be considered as a sphere flattened slightly at the poles; it is, in fact, very nearly of the form of an oblate spheroid, or the surface which would be formed by an ellipse revolving about its minor axis: now there is no reason why we should conclude the attraction of a body such as this to be constant throughout its surface; the precise law according to which the variation takes place we need not inquire, but it seems manifest that such a variation will take place. Thus, to take the two extreme cases, it is not difficult to perceive without investigation that the attraction must be greater at the poles than it is at the equator. But suppose that the earth were truly spherical, and that it revolved slowly upon its axis, as it in fact does, then there would be a centrifugal force upon each particle on its surface, which would be proportional to the distance of the particle from the axis about which the earth revolved.

P. Why so?

T. Let T be the time in which the earth revolves, that is, what we call 24 hours; then if r be the distance of any particle from the axis this particle describes a distance $2\pi r$ in the time T, and therefore its velocity is measured by $\frac{2\pi r}{T}$; hence the measure of the centrifugal force will be $\left(\frac{2\pi r}{T}\right)^2 \div r$, or $\frac{4\pi^2 r}{T^2}$; that is, the centrifugal force will be proportional to r.

P. And this centrifugal force tends to counteract the effect of gravity.

T. Yes; gravity will be diminished by the resolved part of this force in the direction of gravity; if we suppose the earth spherical, gravity will tend towards the centre of the earth, let R be the earth's radius, θ the angle which a line drawn from the centre to the particle in question makes with a line perpendicular to the earth's axis, then you will easily perceive that $r = R\cos\theta$; hence the centrifugal force is measured by $\frac{4\pi^2}{T^2} R\cos\theta$, and resolving this in the direction of the line joining the particle and the earth's centre, we have $\frac{4\pi^2}{T^2} R\cos^2\theta$ for the quantity by which centrifugal force diminishes the force of the earth's attraction.

P. Will this ever be a large quantity?

T. Make $\theta = 0$ so as to give the greatest value possible to it, then the quantity becomes $\frac{4\pi^2 R}{T^2}$; if in this expression you make the following substitutions:

$$R = 20888700,$$
$$T = 24 \times 60 \times 60,$$
$$\pi = 3.14159,$$

and calculate the value, you will then be able to see what proportion the quantity bears to 32, and this will give you a notion of the degree in which the attraction of the earth upon bodies at its surface is modified by centrifugal force.

EXAMINATION UPON CHAPTER VI.

1. Explain what is meant by the term *Centrifugal Force.*

2. A particle is made to revolve in a fixed smooth circular tube; determine the pressure upon the tube.

3. In what time would it be necessary that the earth, considered as a sphere of 4000 miles radius, should revolve, in order that the centrifugal force at the equator should just counteract the earth's attraction; the accelerating force of the earth's attraction being measured by the quantity g?

4. An extensible string stretches one inch for every pound weight suspended from it; find the length of the string when it is made to revolve with a pound weight attached to it in one second, the unstretched length of the string being one yard.

5. In what time must a body weighing n lbs. revolve at the extremity of a string p feet long, in order that the tension of the string may be m lbs.?

6. On the hypothesis of the moon revolving about the earth in a circle, shew how the accelerating force of the earth's attraction at its surface may be deduced; the distance of the moon and the radius of the earth being given.

7. The velocity acquired by a body in falling down a curve in a vertical plane, is that which would have been acquired by the body in falling freely through the same vertical height.

8. A particle runs down the interior of a smooth hemispherical bowl; find the pressure upon the bowl when the particle is at the lowest point.

9. A weight of 1 lb. suspended at one extremity of a string 2 feet in length, the other end of which is fixed, is allowed to fall from a position in which the string is horizontal; find the tension of the string when it is vertical.

10. Explain the method of making a pendulum oscillate in a cycloid.

11. Find the time of oscillation in a cycloid.

12. Find the number of seconds which a pendulum will lose in a day if lengthened by the quantity .2 of an inch; the pendulum being a seconds' pendulum and 39.125 inches in length.

13. Shew how to find the height of a mountain by pendulum-observations.

14. If the seconds' pendulum be 39.125 inches in length, find the accelerating force of gravity.

15. Supposing that in the interior of the earth the intensity of the earth's attraction varies directly as the distance from the centre, find how many seconds per day a pendulum will lose when taken to the bottom of a mine of given depth.

16. Conversely, shew how to find the depth of a mine by means of the pendulum.

17. Suppose that in the figure of page 156 a body is let fall from the highest point A of the cycloid and allowed to run down to C; suppose also that another equal body is allowed to run down a semicircular tube coinciding with the semicircle BC; determine in which case the pressure at C will be the greater.

18. In the figure of page 154, the time required by a particle to move down $P'P$, considered as an inclined plane, will be the same at whatever point of the cycloid P' may be taken.

CHAPTER VII.

PROBLEMS.

THE problems which can be solved by help of the principles explained in the preceding Chapters may be multiplied to any extent in point of number; but our power of solving problems will extend only to the simplest cases of motion, and the following pages will contain only a few additional examples of the same kind as those which have already been given in the examination papers appended to the several Chapters. In the elementary treatise on Statics we had a much wider range; we were able, in fact, to treat of the equilibrium of a rigid body quite generally, with the single exception of confining ourselves to the case of the directions of the forces lying all in one plane; but in treating of Dynamics we were at once restricted to the most simple cases of the motion of a particle in consequence of the student's supposed ignorance of the Differential Calculus, and the motion of a rigid body we can scarcely be said to have entered upon at all. We shall therefore content ourselves with appending some miscellaneous examples of those simple cases of motion which we have been able to discuss, and we have no remarks to make upon the general method of solution in addition to those which will be found in the preceding chapters.

I. Find the velocity of sound by dropping a stone into a well of known depth, omitting all consideration of the resistance of air upon the stone.

II. A body is projected upwards with a velocity of ng feet per second; after how long a time will it be descending with a velocity of pg feet?

III. A falling body is observed to describe in the nth second of its fall a space equal to p times that described in the $\overline{n-1}|^{\text{th}}$: required the whole space described.

IV. Three heavy particles are projected from the same point in the same vertical plane; find the relation between the velocities and directions of projection in order that the three particles may always lie in a straight line, and determine the motion of this line.

V. Find the velocity and direction of projection in order that a projectile may pass horizontally through a given point.

VI. Find the velocity and direction of projection when the path of the projectile in passing through a given point makes an angle of 45^0 with the horizon.

VII. A ball attached to a string is held in such a manner that the string is horizontal; the ball is allowed to fall, and when the string makes an angle of 45^0 with the vertical it is cut; find the latus rectum of the path described by the ball.

VIII. In the preceding problem, determine when the string should be cut in order that the latus rectum of the parabola afterwards described may be the greatest possible, and determine this maximum latus rectum.

IX. Find the distance of a projectile at any given time from a straight line drawn through the point of projection parallel to the direction of the body's motion at the time in question.

X. If P be the point of projection of a projectile, Q its place at the time t, QT a tangent to the path at Q, and PT vertical, then PT is the space through which a body would fall freely in the time t.

XI. If a heavy body be projected in a direction inclined to the horizon, shew that the time of moving between two points at the extremities of a focal chord of the parabolic path is proportional to the product of the velocities of the body at the two points.

XII. A cylinder is made to revolve uniformly about its axis which is vertical, while a body descends under the action of gravity, carrying a pencil which traces a curve on the surface of the cylinder; if the surface of the cylinder be unwrapped, what will be the nature of the curve?

XIII. A body is projected horizontally with a given velocity along a plane inclined at a given angle to the horizon; with what velocity must a body be projected horizontally in free space, so that the parabolas described may be equal?

XIV. If a be the angle of projection of a projectile, T the time which elapses before the body strikes the ground, prove that at the time $\frac{T}{4\sin^2 a}$ the angle which the direction of motion makes with the direction of projection is equal to $\frac{\pi}{2}-a$.

XV. The time of descent of a heavy body from a given point to the centre of a given circle situated vertically below it, is the same as that of its descent to the circumference down an inclined plane which touches the circle.

XVI. If two circles in a vertical plane intersect, the times of descent from the highest points of the circles down inclined planes to either point of intersection will be equal.

XVII. Two given equal circles intersect; given the time of descent from one centre to the other, and the time of descent down the common chord, find the inclination to the vertical of either of these chords.

XVIII. In an inverted parabola the time of descending down any chord from a point in the curve to the vertex is equal to the time of falling freely to a horizontal line which is at a distance below the vertex equal to the latus rectum.

XIX. Find the chord of a vertical circle drawn from the highest point, down which if a body descend it will, after leaving the chord, describe the greatest parabola possible. And determine the value of the latus rectum of this parabola.

XX. If a body be projected from a point in a plane inclined at an angle a to the horizon, the range upon the plane will be greatest when the angle of projection is $45^\circ + \frac{a}{2}$.

XXI. If a body be projected as in the preceding problem, find the angle of projection in order that the focus of the parabolic path may be in the inclined plane.

XXII. The manner of projection being the same, determine the condition in order that the body, supposed perfectly elastic, may after impact rise vertically.

XXIII. If a perfectly elastic body be allowed to fall upon a plane inclined at an angle α to the horizon, and if V be the velocity with which it strikes the plane, then the velocity with which it strikes the plane after the rebound will be $V\dfrac{\sin\left(45^\circ+\dfrac{\alpha}{2}\right)}{\sin\left(45^\circ-\dfrac{\alpha}{2}\right)}$.

XXIV. The latus rectum of the path of a projectile being a, and the horizontal range b, determine the velocity and direction of projection.

XXV. Given the range and the angle of projection, construct the latus rectum.

XXVI. The latus rectum is the height through which the projectile must fall in order to acquire its horizontal velocity.

XXVII. A number of balls of given elasticity A, B, C... are placed in a straight line; A is projected with a given velocity so as to impinge on B; B then impinges on C, and so on; find the masses of the balls B, C... in order that each of the balls A, B, C,... may be at rest after impinging on the next; and find the velocity of the n^{th} ball after the $\overline{n-1}|^{\text{th}}$ has impinged upon it.

XXVIII. A ball of given elasticity is projected in a given direction within a fixed horizontal hoop, so as to rebound from the surface of the hoop; find the velocity after any number of rebounds; shew also that the ball will eventually move round upon the surface of the hoop with a certain constant velocity, and find this velocity.

XXIX. What will take place if in the preceding problem the ball be considered perfectly elastic?

XXX. Two equal balls A and B are moving with given velocities in the same plane in directions at right angles to each other, and the line joining their centres at the instant of impact is in the direction of A's motion. Determine their motions after impact, supposing them smooth and inelastic.

XXXI. Solve the preceding problem supposing the balls to be elastic.

XXXII. A heavy elastic particle slides down an inclined plane of given height under the action of gravity, and impinging upon a hard horizontal plane rebounds; find the inclination of the former plane in order that the range upon the latter may be the greatest possible.

XXXIII. Three equal elastic balls are moving in the same direction with velocities which are proportional to 3, 2, 1, respectively, and the distances between them at a given time were equal; find the velocities after impact, and shew that they continue to be in arithmetical progression.

XXXIV. Consider the case of perfect elasticity. What will be the common difference of the arithmetical progression in this case?

XXXV. Generalize Prob. XXXIII., by taking the velocities of the balls to be $V+a$, V, and $V-a$, and prove that in this case the velocities after impact form an arithmetical progression.

XXXVI. A smooth tube of uniform bore is bent into the form of a circular arc greater than a semicircle, and placed in a vertical plane with its open ends upwards and in the same horizontal line. Find the velocity with which a ball fitting the tube must be projected along the interior from the lowest point, in order that it may pass out at one end and re-enter at the other.

XXXVII. Why is the condition, "greater than a semicircle," introduced into the preceding problem? What would be the motion if the arc were exactly a semicircle? and what if the arc were less?

XXXVIII. Two elastic particles are allowed to fall at the same instant into the extremities of a semicircular tube placed as in the preceding problem; shew what the nature of the motion will be, and determine the height to which either of the particles will rise after any number of impacts.

XXXIX. Two equal balls, elasticity e, start at the same instant with equal velocities from the opposite angles of a square along the sides, and impinge; determine the angle between the directions of their *motion* after impact.

XL. An imperfectly elastic ball is projected from a point between two vertical planes, the plane of motion being perpendicular to both; shew that the arcs described between the rebounds are portions of parabolas whose latera recta are in geometrical progression.

XLI. Find the time which elapses between the first and twenty-first rebound, in the preceding problem.

XLII. A body of given elasticity e is projected along a horizontal plane from the middle point of one of the sides of an isosceles right-angled triangle, so as after reflexion at the hypothenuse and remaining side to return to the same point; shew that the cotangents of the angles of reflexion are $e+1$ and $e+2$ respectively.

XLIII. When an elastic ball rebounds any number of times on a horizontal plane, the curvilinear paths of the body between successive rebounds are all portions of the same parabola.

XLIV. When a weight W draws up another weight $2W'$ by means of a string and pully, the tension is found to be T, when W draws up W' it is found to be T'; find W in terms of T and T'.

XLV. Two weights are connected by a string passing over a small pully; in the beginning of the motion one of the greater weights is higher than the other by a given number of feet; when the two are in the same horizontal line they are suddenly set free from the string; determine the subsequent motion. For example, determine the vertical distance between the weights at a given time.

XLVI. A bead running upon a fine thread the extremities of which are fixed describes an ellipse in a plane passing through the extremities, under the action of no external force; prove that the velocity of the bead will be constant.

XLVII. A railway train is going smoothly along a curve of 500 yards radius at the rate of 30 miles per hour; find at what angle a plumbline hanging in one of the carriages will be inclined to the vertical.

The weight suspended may be regarded as a particle kept in equilibrium by the tension of the line, its own weight, and the centrifugal force. See page 140.

XLVIII. A pendulum which oscillates seconds at one place is carried to a place where it gains two minutes in 24 hours; compare the force of gravity at the two places.

XLIX. A seconds' pendulum was too long on a given day by a small quantity a, it was then over-corrected so as to be too short by a during the next day: shew that the number of minutes gained in the two days was $1080 \frac{a^2}{L^2}$ nearly, L being the length of the seconds' pendulum.

L. A pendulum is found to make 640 vibrations at the equator in the same time as it makes 641 at Greenwich; if a string hanging vertically can just sustain 80 pounds at Greenwich, how many such pounds can the same string sustain at the equator?

THE END.

New Works and New Editions

FOLIORUM SILVULA: a Selection of Passages for Translation into Greek and Latin Verse, mainly from the University and College Examination Papers; Edited by H. A. HOLDEN, M.A., Vice Principal of Cheltenham College, Fellow and late Assistant Tutor of Trinity College, Cambridge.
Post 8vo. *7s.*

By the same,

FOLIORUM CENTURIÆ. Selections for Translation into Latin and Greek Prose, chiefly from the University and College Examination Papers. Post 8vo. *7s.*

GREEK TRAGIC SENARII, Progressive Exercises in, followed by a Selection from the Greek Verses of Shrewsbury School, and prefaced by a short Account of the Iambic Metre and Style of Greek Tragedy. By the Rev. B. H. KENNEDY, D.D., Prebendary of Lichfield, and Head-Master of Shrewsbury School. For the use of Schools and Private Students. Second Edition, altered and revised. 8vo. *8s.*

PASSAGES in PROSE and VERSE from ENGLISH Authors for Translation into Greek and Latin; together with selected Passages from Greek and Latin Authors for Translation into English: forming a regular course of Exercises in Classical Composition. By the Rev. H. ALFORD, M.A., Vicar of Wymeswold, Leicestershire, late Fellow of Trinity College, Cambridge. 8vo. *6s.*

An HISTORICAL & EXPLANATORY TREATISE on the BOOK of COMMON PRAYER. By the Rev. W. G. HUMPHRY, B.D., late Fellow of Trinity College, Cambridge. Examining Chaplain to the Lord Bishop of London.
Post 8vo. *7s. 6d.*

Published by J. Deighton, Cambridge.

A TREATISE on HANNIBAL'S PASSAGE of the ALPS, in which his Route is traced over the Little Mont Cenis. Illustrated with Maps. By ROBERT ELLIS, B.D., Fellow of St. John's College, Cambridge.

PHRASEOLOGICAL & EXPLANATORY NOTES on the HEBREW TEXT of the Book of GENESIS. By the Rev. T. PRESTON, M.A., Fellow of Trinity College, Cambridge. Crown 8vo. 7*s.* 6*d.*

A TRANSLATION of the EPISTLES of CLEMENT of Rome, POLYCARP, and IGNATIUS; and of the Apologies of JUSTIN MARTYR and TERTULLIAN: with an Introduction and Brief Notes illustrative of the Ecclesiastical History of the First Two Centuries. By the Rev. T. CHEVALLIER, B.D., late Fellow and Tutor of St. Catharine's Hall. New Edition. 8vo. 1?*s.*

A HISTORY of the ARTICLES of RELIGION. Including, among other Documents, the X Articles (1536), the XIII Articles (1538), the XLII Articles (1552), the XI Articles (1559), the XXXIX Articles (1562 and 1571), the Lambeth Articles (1595), the Irish Articles (1615), with Illustrations from the Symbolical Books of the Roman and Reformed Communions, and from other contemporary sources. By C. HARDWICK, M.A., Fellow and Chaplain of St. Catharine's Hall. 8vo. 10*s.* 6*d.*

PALMER'S ORIGINES LITURGICÆ, an Analysis of; or, Antiquities of the English Ritual; and of his Dissertation on Primitive Liturgies: for the use of Students at the Universities, and Candidates for Holy Orders, who have read the Original Work. By W. BEAL, LL.D., F.R.S., Vicar of Brooke, Norfo[redacted] 12mo. 3*s.* 6*d.*